Text-Book on Trigonometry

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TEXT-BOOK ON TRIGONOMETRY

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PREFACE

This book is written to meet the requirements of the B.A. and B.Sc. students of Indian Universities. Attempt has been made to make the subject easy and understandable to the average student. Expansion of fundamental principles is made clear and concise, and unnecessary details have been avoided. At the same time the treatment is rigorous and up-to-date.

The number of examples is fairly large, and care has been taken to include all varieties of questions. They have been taken mostly from examination papers so that the student may be familiar with the types of questions which are usually set at the examinations. Some typical examples have been solved by way of illustrating the principles.

We are grateful to our teacher Dr. Gorakh Prasad, D.Sc., of the University of Allahabad, for his valuable advice and keen interest in the preparation of this book.

Corrections and suggestions will be thankfully received.

R. S. Varma

January, 1951 K. S. Shukla

PREFACE TO THE SEVENTH EDITION

In preparing this edition, we have taken the opportunity of reading the book and verifying the questions once again. Various additions and modifications made here and there will, it is hoped, add to the usefulness of the book. A list of trigonometrical formulae has been given in the beginning of the book for ready reference to them. Special care has been taken to make the book free from errors.

May, 1969 Authors

CONTENTS

TRIGONOMETRICAL FORMULAE

1. $\pi = 3{\cdot}14159$

   1 radian $= 57{\cdot}2957795 \ldots$ degrees

   $= 3438$ minutes approx.

   $= 206265$ seconds approx.

2. $\sin^2\theta + \cos^2\theta = 1$

   $\sec^2\theta = 1 + \tan^2\theta$

   $\operatorname{cosec}^2\theta = 1 + \cot^2\theta.$

3. (i)

	$\theta=0$	15°	18°	30°	36°	45°
$\sin\theta$	0	$\frac{\sqrt{6}-\sqrt{2}}{4}$	$\frac{\sqrt{5}-1}{4}$	$\frac{1}{2}$	$\frac{\sqrt{(10-2\sqrt{5})}}{4}$	$\frac{1}{\sqrt{2}}$
$\cos\theta$	1	$\frac{\sqrt{6}+\sqrt{2}}{4}$	$\frac{\sqrt{(10+2\sqrt{5})}}{4}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{5}+1}{4}$	$\frac{1}{\sqrt{2}}$
$\tan\theta$	0	$2-\sqrt{3}$	$\frac{\sqrt{5}-1}{\sqrt{(10+2\sqrt{5})}}$	$\frac{1}{\sqrt{3}}$	$\frac{\sqrt{(10-2\sqrt{5})}}{\sqrt{5}+1}$	1

(ii)

	$\theta=54^\circ$	60°	72°	75°	90°
$\sin\theta$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{(10+2\sqrt{5})}}{4}$	$\frac{\sqrt{6}+\sqrt{2}}{4}$	1
$\cos\theta$	$\frac{\sqrt{(10-2\sqrt{5})}}{4}$	$\frac{1}{2}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{6}-\sqrt{2}}{4}$	0
$\tan\theta$	$\frac{\sqrt{5}+1}{\sqrt{(10-2\sqrt{5})}}$	$\sqrt{3}$	$\frac{\sqrt{(10+2\sqrt{5})}}{\sqrt{5}-1}$	$2+\sqrt{3}$	∞

4. $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$.

$\sin(90^\circ - \theta) = \cos\theta$, $\cos(90^\circ - \theta) = \sin\theta$.

$\sin(90^\circ + \theta) = \cos\theta$, $\cos(90^\circ + \theta) = -\sin\theta$.

$\sin(180^\circ - \theta) = \sin\theta$, $\cos(180^\circ - \theta) = -\cos\theta$.

$\sin(180^\circ + \theta) = -\sin\theta$, $\cos(180^\circ + \theta) - \cos\theta$.

sine (+) cosine (−)	sine (+) cosine (+)
sine (−) cosine (−)	sine (−) cosine (+)

5. If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n\alpha$;

if $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha$;

if $\tan\theta = \tan\alpha$, then $\theta = n\pi + \alpha$;

n being zero or any integer, positive or negative.

6. $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$,

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$.

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}.$$

$$\cot(\alpha \pm \beta) = \frac{\cot\alpha\cot\beta \mp 1}{\cot\beta \pm \cot\alpha}.$$

7. $$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}.$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}.$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}.$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}.$$

Conversely—

$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$

$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$

$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$

$\sin\alpha\sin\beta = -\frac{1}{2}[\cos(\alpha+\beta) - \cos(\alpha-\beta)]$

8. $\sin 2\theta = 2 \sin \theta \cos \theta = \dfrac{2 \tan \theta}{1+\tan^2 \theta}$.

$\cos 2\theta = \cos^2 \theta - \sin^2 \theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta$

$= \dfrac{1-\tan^2 \theta}{1+\tan^2 \theta}$.

$\tan 2\theta = \dfrac{2 \tan \theta}{1-\tan^2 \theta}$, $\cot 2\theta = \dfrac{\cot^2 \theta - 1}{2 \cot \theta}$.

Also

$\sin^2 \theta = \frac{1}{2}(1-\cos 2\theta)$

$\cos^2 \theta = \frac{1}{2}(1+\cos 2\theta)$.

9. $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$.

$\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$.

$\tan 3\theta = \dfrac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$.

10. $\tan n\theta = \dfrac{{}^nC_1 \tan \theta - {}^nC_3 \tan^3 \theta + \dots}{1 - {}^nC_2 \tan^2 \theta + \dots}$.

$\tan(\theta_1+\theta_2+\dots+\theta_n) = \dfrac{s_1 - s_3 + s_5 - \dots}{1 - s_2 + s_4 - \dots}$,

where s_r denotes the sum of the products of $\tan \theta_1$, $\tan \theta_2$, $\tan \theta_3$, ..., $\tan \theta_n$ taken r at a time.

11. $\log_e xy = \log_e x + \log_e y$.

$\log_e (x/y) = \log_e x - \log_e y$.

$\log_e x^y = y \log_e x$.

$\log_x y = \dfrac{\log y}{\log x}$.

$\log 1 = 0$, $\log 0 = -\infty$, $\log \infty = \infty$.

Also

$e^0 = 1$, $e^{-\infty} = 0$, $e^{\infty} = \infty$.

12. If a, b, c be the sides, A, B, C the opposite angles, Δ the area, R, r, r_1 the radii of the circumscribed circle,

inscribed circle and escribed circle opposite A, of any triangle ABC, and $a+b+c=2s$, then

(i) $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=2R.$

(ii) $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$,

$b\cos C+c\cos B=a.$

(iii) $\sin\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{bc}}$, $\cos\dfrac{A}{2}=\sqrt{\dfrac{s(s-a)}{bc}}$,

$\tan\dfrac{A}{2}=\sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$,

$\sin A=\dfrac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}=\dfrac{2\triangle}{bc}.$

(iv) $\triangle=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B=\frac{1}{2}ab\sin C$

$=\sqrt{s(s-a)(s-b)(s-c)}.$

$R=\dfrac{a}{2\sin A}=\dfrac{b}{2\sin B}=\dfrac{c}{2\sin C}=\dfrac{abc}{4\triangle}$,

$r=\dfrac{\triangle}{s}=4R\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

$=(s-a)\tan\dfrac{A}{2}=(s-b)\tan\dfrac{B}{2}=(s-c)\tan\dfrac{C}{2}.$

$r_1=\dfrac{\triangle}{s-a}=4R\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}=s\tan\dfrac{A}{2}.$

13. $\sin^{-1}x+\cos^{-1}x=\frac{1}{2}\pi.$

$\tan^{-1}x+\cot^{-1}x=\frac{1}{2}\pi.$

$\tan^{-1}x-\tan^{-1}y=\tan^{-1}\dfrac{x-y}{1+xy}.$

$\cot^{-1}x-\cot^{-1}y=\cot^{-1}\dfrac{xy+1}{y-x}.$

14. $(\cos\theta\pm i\sin\theta)^n=\cos n\theta\pm i\sin n\theta$,
where n is any rational number, positive or negative.

15. $\sin\theta=\theta-\dfrac{\theta^3}{3!}+\dfrac{\theta^5}{5!}-\ldots+(-)^{n-1}\dfrac{\theta^{2n-1}}{(2n-1)!}+\ldots$

$\cos\theta=1-\dfrac{\theta^2}{2!}+\dfrac{\theta^4}{4!}-\ldots+(-)^{n-1}\dfrac{\theta^{2n-2}}{(2n-2)!}+\ldots$

$\tan\theta=\theta+\frac{1}{3}\theta^3+\frac{2}{15}\theta^5+\frac{17}{315}\theta^7+\ldots$

16. $\sin\theta=\dfrac{e^{\theta i}-e^{-\theta i}}{2i}$, $\cos\theta=\dfrac{e^{\theta i}+e^{-\theta i}}{2}$,

$e^{\theta i}=\cos\theta+i\sin\theta$, $e^{-\theta i}=\cos\theta-i\sin\theta$.

17. $\log_e(x+iy)=\log_e\sqrt{(x^2+y^2)}+i\tan^{-1}(y/x)$.

$\mathrm{Log}_e(x+iy)=2n\pi i+\log(x+iy)$.

$u^v=e^{v\,\mathrm{Log}_e u}$.

18. (*i*) $\sinh\theta=\theta+\dfrac{\theta^3}{3!}+\dfrac{\theta^5}{5!}+\ldots$

$=\dfrac{e^{\theta}-e^{-\theta}}{2}$,

$\cosh\theta=1+\dfrac{\theta^2}{2!}+\dfrac{\theta^4}{4!}+\ldots$

$=\dfrac{e^{\theta}+e^{-\theta}}{2}$,

(ii) $e^{\theta}=\cosh\theta+\sinh\theta$,

$e^{-\theta}=\cosh\theta-\sinh\theta$.

(iii) $\sinh(-\theta)=-\sinh\theta$,

$\cosh(-\theta)=\cosh\theta$.

(iv) $\sin(i\theta)=i\sinh\theta$ | $\sinh(i\theta)=i\sin\theta$

$\cos(i\theta)=\cosh\theta$ | $\cosh(i\theta)=\cos\theta$

$\tan(i\theta)=i\tanh\theta$. | $\tanh(i\theta)=i\tan\theta$.

(v) $\cosh^2\theta-\sinh^2\theta=1$

$\tanh^2\theta+\mathrm{sech}^2\theta=1$

$\coth^2\theta-\mathrm{cosech}^2\theta=1$.

(vi) $\sinh(u \pm v) = \sinh u \cosh v \pm \cosh u \sinh v.$

$\cosh(u \pm v) = \cosh u \cosh v \pm \sinh u \sinh v.$

$$\tanh(u \pm v) = \frac{\tanh u \pm \tanh v}{1 \pm \tanh u \tanh v}.$$

$$\coth(u \pm v) = \frac{\coth u \coth v \pm 1}{\coth v \pm \coth u}.$$

(vii) $\sinh u + \sinh v = 2 \sinh \frac{u+v}{2} \cosh \frac{u-v}{}.$

$$\sinh u - \sinh v = 2 \cosh \frac{u+v}{2} \sinh \frac{u-v}{2}.$$

$$\cosh u + \cosh v = 2 \cosh \frac{u+v}{2} \cosh \frac{u-v}{2}.$$

$$\cosh u - \cosh v = 2 \sinh \frac{u+v}{2} \sinh \frac{u-v}{2}.$$

Conversely

$\sinh u \cosh v = \frac{1}{2} [\sinh(u+v) + \sinh(u-v)]$

$\cosh u \sinh v = \frac{1}{2} [\sinh(u+v) - \sinh(u-v)]$

$\cosh u \cosh v = \frac{1}{2} [\cosh(u+v) + \cosh(u-v)]$

$\sinh u \sinh v = \frac{1}{2} [\cosh(u+v) - \cosh(u-v)]$

(viii) $\sinh 2\theta = 2 \sinh \theta \cosh \theta$

$\cosh 2\theta = \cosh^2 \theta + \sinh^2 \theta = 2\cosh^2 \theta - 1 = 2 \sinh^2 \theta + 1.$

Also

$\sinh^2 \theta = \frac{1}{2} (\cosh 2\theta - 1)$

$\cosh^2 \theta = \frac{1}{2} (\cosh 2\theta + 1).$

(ix) $\sinh^{-1} x = \log [x + \sqrt{(x^2+1)}]$

$\cosh^{-1} x = \log [x + \sqrt{(x^2-1)}]$

$$\tanh^{-1} x = \frac{1}{2} \log \frac{1+x}{1-x}.$$

19. (i) $(2 \cos \theta)^n = 2[\cos n\theta + {}^nC_1 \cos(n-2)\theta$
$+ {}^nC_2 \cos(n-4)\theta + \ldots],$
n being a positive integer.

(ii) $$(2i\sin\theta)^n = 2\Big[\cos n\theta - {}^nC_1\cos(n-2)\theta + {}^nC_2\cos(n-4)\theta - \dots\Big],$$

n being an even positive integer.

$$= 2i\Big[\sin n\theta - {}^nC_1\sin(n-2)\theta + {}^nC_2\sin(n-4)\theta - \dots\Big]$$

n being an odd positive integer.

20. When n is a positive integer,

(i) $$\frac{\sin n\theta}{\sin\theta} = (2\cos\theta)^{n-1} - (n-2)(2\cos\theta)^{n-3} + \frac{(n-3)(n-4)}{2!}(2\cos\theta)^{n-5} - \dots$$
$$+ (-1)^r\frac{(n-r-1)(n-r-2)\dots(n-2r)}{r!}(2\cos\theta)^{n-2r-1} + \dots$$

(ii) $$2\cos n\theta = (2\cos\theta)^n - n(2\cos\theta)^{n-2} + \frac{n(n-3)}{2!}(2\cos\theta)^{n-4} - \dots$$
$$+ (-1)^r\frac{n(n-r-1)(n-r-2)\dots(n-2r+1)}{r!}(2\cos\theta)^{n-2r} + \dots$$

21. (*a*) When n is an even positive integer

$$\frac{\sin n\theta}{\cos\theta} = n\sin\theta - \frac{n(n^2-2^2)}{3!}\sin^3\theta + \frac{n(n^2-2^2)(n^2-4^2)}{5!}\sin^5\theta - \dots + (-1)^{n/2+1}(2\sin\theta)^{n-1};$$

$$\frac{\sin n\theta}{\sin\theta} = (-1)^{-n/2-1}\Big[n\cos\theta - \frac{(n^2-2^2)}{3!}\cos^3\theta + \frac{n(n^2-2^2)(n^2-4^2)}{5!}\cos^5\theta - \dots + (-1)^{n/2+1}(2\cos\theta)^{n-1}\Big].$$

(ii) $$\cos n\theta = 1 - \frac{n^2}{2!}\sin^2\theta + \frac{n^2(n^2-2^2)}{4!}\sin^4\theta - \ldots$$

$$= (-1)^{n/2}\left[1 - \frac{n^2}{2!}\cos^2\theta + \frac{n^2(n^2-2^2)}{4!}\cos^4\theta - \ldots\right].$$

(b) When n is an odd positive integer

(i) $$\frac{\sin n\theta}{\sin\theta} = n - \frac{n(n^2-1^2)}{3!}\sin^2\theta - \frac{n(n^2-1^2)(n^2-3^2)}{5!}\sin^4\theta - \ldots$$

$$= (-)^{-(n-1)/2}\left[1 - \frac{n^2-1^2}{2!}\cos^2\theta + \frac{(n^2-1^2)(n^2-3^2)}{4!} \times \cos^4\theta - \ldots\right].$$

(ii) $$\frac{\cos n\theta}{\cos\theta} = 1 - \frac{n^2-1^2}{2!}\sin^2\theta + \frac{(n^2-1^2)(n^2-3^2)}{4!}\sin^4\theta - \ldots$$

$$= (-1)^{-(n-1)/2}\left[n - \frac{n(n^2-1^2)}{3!}\cos^2\theta + \frac{n(n^2-1^2)(n^2-3^2)}{5!}\cos^4\theta - \ldots\right].$$

22. $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$, where $-1 \leqslant x \leqslant 1$.

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \ldots.$$

23. (i) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots$ to n terms

$$= \frac{\sin(n\beta/2)}{\sin(\beta/2)}\sin\left(\alpha + \frac{n-1}{2}\beta\right).$$

$$= \frac{\sin\dfrac{n\ \text{diff}}{2}}{\sin\dfrac{\text{diff}}{2}}\sin\left[\frac{\text{first angle} + \text{last angle}}{2}\right].$$

(ii) $\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots$ to n terms

$$=\frac{\sin(n\beta/2)}{\sin(\beta/2)}\cos\left(\alpha+\frac{n-1}{2}\beta\right)$$

$$=\frac{\sin\dfrac{n\text{ diff}}{2}}{\sin\dfrac{\text{diff}}{2}}\cos\left[\frac{\text{first angle}+\text{last angle}}{2}.\right]$$

24. (i) $x^n-1=(x^2-1)\prod_{r=1}^{n/2-1}[x^2-2x\cos(2r\pi/n)+1],$

when n is even;

$$=(x-1)\prod_{r=1}^{(n-1)/2}[x^2-2x\cos(2r\pi/n)+1],$$

when n is odd.

(ii) $x^n+1=\prod_{r=0}^{n/2-1}[x^2-2x\cos\{(2r+1)\pi/n\}+1],$

when n is even;

$$=(x+1)\prod_{r=0}^{(n-3)/2}[x^2-2x\cos\{(2r+1)\pi/n\}+1],$$

when n is odd.

(iii) $x^{2n}-2x^n\cos n\theta+1=\prod_{r=0}^{n-1}[x^2-2x\cos(\theta+2r\pi/n)+1].$

25. (i) $\sin\theta=\theta\prod_{r=1}^{\infty}\left(1-\frac{\theta^2}{r^2\pi^2}\right).$

$$\cos\theta=\prod_{r=1}^{\infty}\left[1-\frac{4\theta^2}{(2r-1)^2\pi^2}\right].$$

(ii) $\sinh\theta=\theta\prod_{r=1}^{\infty}\left(1+\frac{\theta^2}{r^2\pi^2}\right)$

$$\cosh\theta=\prod_{r=1}^{\infty}\left[1+\frac{4\theta^2}{(2r-1)^2\pi^2}\right].$$

TRIGONOMETRY

CHAPTER I

INVERSE TRIGONOMETRICAL FUNCTIONS

1·1. Definition. The equation $\sin\theta = x$ means that x is the sine of the angle θ. We now introduce the notation $\sin^{-1} x$ (read *sine inverse* x), and say that the relation

$$\sin^{-1} x = \theta$$

means that θ is the angle whose sine is x. Thus $\sin^{-1} x$ is an angle. Similarly, we may define $\cos^{-1} x$ $\tan^{-1} x$, $\cot^{-1} x$, $\sec^{-1} x$ and $\mathrm{cosec}^{-1} x$. These are called inverse circular functions.

The student must not confuse $\sin^{-1} x$ with $(\sin x)^{-1}$, i.e., $\dfrac{1}{\sin x}$.

1·11. Principal value of an inverse function. It is easy to see that the inverse circular functions are many-valued functions. For, as is well known, if θ is the least positive angle whose sine is x then all angles given by $n\pi + (-1)^n \theta$ have their sines equal to x. So we see that

$$\sin^{-1} x = n\pi + (-1)^n \theta,$$

where n may have any integral value.

The *principal value* of an inverse function is that value of the inverse function which is numerically the smallest. It may be positive or negative. In case there are two values, one positive and the

2 T

other negative, which are numerically equal, the principal value is the positive value.

In this book, unless the contrary is stated, any inverse function under consideration should be understood to have its principal value. Thus

$$\sin^{-1}(\sqrt{3}/2)=\tfrac{1}{3}\pi,\ \cos^{-1}(\sqrt{3}/2)=\tfrac{1}{6}\pi.$$

1·2. Relations between inverse functions. We have seen that $\sin^{-1} x=\theta$ means that $\sin\theta=x$. But when $\sin\theta=x$,

$$\operatorname{cosec}\theta=1/x,$$

which may be written as

$$\operatorname{cosec}^{-1}(1/x)=\theta.$$

It follows that

$$\sin^{-1} x=\operatorname{cosec}^{-1}(1/x).$$

Similarly, $\cos^{-1} x=\sec^{-1}(1/x)$,

and $\tan^{-1} x=\cot^{-1}(1/x)$.

Also, substituting $\sin\theta$ for x in $\sin^{-1} x=\theta$, we get

$$\theta=\sin^{-1}(\sin\theta);$$

and substituting $\sin^{-1} x$ for θ in $x=\sin\theta$, we get

$$x=\sin(\sin^{-1} x).$$

Similarly, $\theta=\cos^{-1}(\cos\theta)=\tan^{-1}(\tan\theta)$, etc. and $x=\cos(\cos^{-1} x)=\tan(\tan^{-1} x)$, etc.

Moreover, when $\sin\theta=x$,

$$\cos\theta=\sqrt{(1-x^2)} \text{ and } \tan\theta=x/\sqrt{(1-x^2)}.$$

Hence

$$\theta=\sin^{-1} x=\cos^{-1}\sqrt{(1-x^2)}=\tan^{-1}\frac{x}{\sqrt{(1-x^2)}}.$$

We can further show that each one of these is also equal to

$$\cot^{-1}\frac{\sqrt{(1-x^2)}}{x}=\sec^{-1}\frac{1}{\sqrt{(1-x^2)}}=\operatorname{cosec}^{-1}\frac{1}{x}.$$

These relations may be easily written down by the help of a right-angled triangle. For in the triangle ABC, right-angled at C, we know that

$$\sin\angle ABC=\frac{AC}{AB},\ \cos\angle ABC=\frac{BC}{AB},$$

$$\tan\angle ABC=\frac{AC}{BC},\ \text{etc.}$$

Therefore

$$ABC=\sin^{-1}\frac{AC}{AB}=\cos^{-1}\frac{BC}{AB}=\tan^{-1}\frac{AC}{BC},\ \text{etc.} \qquad (1)$$

Suppose that $\angle ABC=\sin^{-1}x$. Then assuming $AC=x$ and $AB=1$, so that $BC=\sqrt{(1-x^2)}$, we have from (1) that

$$\sin^{-1}x=\cos^{-1}\sqrt{(1-x^2)}=\tan^{-1}\frac{x}{\sqrt{(1-x^2)}},\ \text{etc.}$$

Similarly, if $\angle ABC=\tan^{-1}y$, then assuming $AC=y$ and $BC=1$, so that $AB=\sqrt{(1+y^2)}$, we have from (1) that

$$\tan^{-1}y=\sin^{-1}\frac{y}{\sqrt{(1+y)^2}}=\cos^{-1}\frac{1}{\sqrt{(1+y^2)}}\ \text{etc.}$$

1·3. Addition formulae. Any relation between trigonometrical functions can be expressed also by means of the inverse notation. For example, we know that

$$\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}.$$

This may be written as

$$2\theta=\tan^{-1}\frac{2\tan\theta}{1-\tan^2\theta}.$$

If we put $\tan\theta = x$, so that $\tan^{-1} x = \theta$, this gives

$$2\tan^{-1} x = \tan^{-1}\frac{2x}{1-x^2}.$$

This is a particular case of the general formula

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}.$$

To prove this, suppose that

$$\tan^{-1} x = \alpha, \text{ and } \tan^{-1} y = \beta,$$

so that $\tan\alpha = x$ and $\tan\beta = y$.

Then $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} = \dfrac{x+y}{1-xy}.$

Therefore $\alpha + \beta = \tan^{-1}\dfrac{x+y}{1-xy},$

which gives the above formula on restoring the values of α and β.

Similarly, we can show that

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\frac{x-y}{1+xy}.$$

1·4. Other formulae. The following formulae may be easily shown to be true:

(i) $\sin^{-1} x + \cos^{-1} x = \frac{1}{2}\pi$,

(ii) $\tan^{-1} x + \cot^{-1} x = \frac{1}{2}\pi$,

(iii) $\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{1}{2}\pi$.

To show the first of these, say, let $x = \sin\theta$.

Then since $\sin\theta = \cos(\frac{1}{2}\pi - \theta)$, we have

$$\sin^{-1} x = \theta.$$

and $\cos^{-1} x = \frac{1}{2}\pi - \theta.$

Hence $$\sin^{-1} x+\cos^{-1} x=\tfrac{1}{2}\pi.$$

Ex. 1. Show that $3 \sin^{-1} x=\sin^{-1}(3x-4x^3)$.

We know that

$$\sin 3\alpha=3 \sin \alpha-4 \sin^3 \alpha.$$

If we take $\sin \alpha=x$, we have

$$3\alpha=\sin^{-1}(3 \sin \alpha-4 \sin^3 \alpha),$$

or $$3 \sin^{-1} x=\sin^{-1}(3x-4x^3).$$

Ex. 2. Prove that

$$2 \tan^{-1}\left\{\sqrt{\left(\frac{a-b}{a+b}\right)} \tan \tfrac{1}{2}x\right\}=\cos^{-1}\left(\frac{b+a \cos x}{a+b \cos x}\right).$$

[*Banaras*, 1947; *Jammu* & *Kashmir*, 1953]

Using the formula

$$2 \tan^{-1} y=\tan^{-1} \frac{2y}{1-y^2},$$

we have

$$2 \tan^{-1}\left\{2\sqrt{\left(\frac{a-b}{a+b}\right)} \tan \tfrac{1}{2}x\right\}$$

$$=\tan^{-1}\left\{\sqrt{\left(\frac{a-b}{a+b}\right)} \tan \tfrac{1}{2}x\Big/\left(1-\frac{a-b}{a+b} \tan^2 \tfrac{1}{2}x\right)\right\}$$

$$=\tan^{-1}[\{2\surd(a^2-b^2)\sin \tfrac{1}{2}x \cos \tfrac{1}{2}x\}/\{a(\cos^2 \tfrac{1}{2}x-\sin^2 \tfrac{1}{2}x) +b(\cos^2 \tfrac{1}{2}x+\sin^2 \tfrac{1}{2}x)\}]$$

$$=\tan^{-1}[\{\surd(a^2-b^2)\sin x\}/(a \cos x+b)].$$

But $$\{\surd(a^2-b^2) \sin x\}^2+(a \cos x+b)^2$$

$$=(a^2-b^2) \sin^2 x+(a^2 \cos^2 x+2ab \cos x+b^2)$$

$$=a^2+2ab \cos x+b^2 \cos^2 x$$

$$=(a+b \cos x)^2.$$

Therefore, on using the formula

$$\tan^{-1} y=\cos^{-1} \frac{1}{\surd(1+y^2)},$$

the above expression is equal to

$$\cos^{-1}\left(\frac{b+a \cos x}{a+b \cos x}\right).$$

ALTERNATIVE METHOD.

Put $\theta = 2\tan^{-1}\left\{\sqrt{\left(\dfrac{a-b}{a+b}\right)}\tan\tfrac{1}{2}x\right\}$.

Then $\tan\tfrac{1}{2}\theta = \sqrt{\left(\dfrac{a-b}{a+b}\right)}\tan\tfrac{1}{2}x$,

which on squaring gives

$$\frac{\sin^2\frac{1}{2}\theta}{\cos^2\frac{1}{2}\theta} = \frac{(a-b)\sin^2\frac{1}{2}x}{(a+b)\cos^2\frac{1}{2}x}.$$

Applying Componendo and Dividendo, we have

$$\frac{\cos^2\frac{1}{2}\theta - \sin^2\frac{1}{2}\theta}{\cos^2\frac{1}{2}\theta + \sin^2\frac{1}{2}\theta} = \frac{(a+b)\cos^2\frac{1}{2}x - (a-b)\sin^2\frac{1}{2}x}{(a+b)\cos^2\frac{1}{2}x + (a-b)\sin^2\frac{1}{2}x}$$

or
$$\cos\theta = \frac{b+a\cos x}{a+b\cos x}.$$

Therefore $\theta = 2\tan^{-1}\left\{\sqrt{\left(\dfrac{a-b}{a+b}\right)}\tan\tfrac{1}{2}x\right\} = \cos^{-1}\left(\dfrac{b+a\cos x}{a+b\cos x}\right)$.

Ex. 3. Solve the equation

$$\tan^{-1}\frac{1}{1+2x} + \tan^{-1}\frac{1}{4x+1} = \tan^{-1}\frac{2}{x^2}. \qquad [\textit{Agra}, 1947]$$

Since $\tan^{-1}y + \tan^{-1}z = \tan^{-1}\dfrac{y+z}{1-yz}$,

we have

$$\tan^{-1}\frac{1}{1+2x} + \tan^{-1}\frac{1}{1+4x}$$
$$= \tan^{-1}\left[\left(\frac{1}{1+2x}+\frac{1}{1+4x}\right)\Big/\left\{1-\frac{1}{(1+2x)(1+4x)}\right\}\right]$$
$$= \tan^{-1}\frac{2(1+3x)}{2x(3+4x)} = \tan^{-1}\frac{1+3x}{x(3+4x)}.$$

Hence

$$\tan^{-1}\frac{1+3x}{x(3+4x)} = \tan^{-1}\frac{2}{x^2},$$

i.e.,
$$\frac{1+3x}{x(3+4x)} = \frac{2}{x^2},$$

or $$x^2+3x^3=6x+8x^2,$$

or $$x(3x^2-7x-6)=0,$$

or $$x(x-3))(3x+2)=0.$$

Therefore $$x=0,\ 3,\ \text{or}\ -\tfrac{2}{3}.$$

Ex. 4. Solve

$$\sin^{-1}x+\sin^{-1}y=\tfrac{2}{3}\pi.$$

$$\cos^{-1}x-\cos^{-1}y=\tfrac{1}{3}\pi. \qquad \text{[}\textit{Gauhati}, 1950\text{]}$$

From § 1·4, $\sin^{-1}x=\frac{1}{2}\pi-\cos^{-1}x$ and $\sin^{-1}y=\frac{1}{2}\pi-\cos^{-1}y$. Therefore the given equations may be written as

$$\cos^{-1}x+\cos^{-1}y=\tfrac{1}{3}\pi. \qquad (1)$$

$$\cos^{-1}x-\cos^{-1}y=\tfrac{1}{3}\pi. \qquad (2)$$

Adding (1) and (2), we get $\cos^{-1}x=\frac{1}{3}\pi$. Therefore $x=\frac{1}{2}$.

Subtracting (2) from (1), we get $\cos^{-1}y=0$. Therefore $y=1$.

Hence $x=1/2, y=1$.

Examples

1. Show that

(i) $2\sin^{-1}x=\sin^{-1}\{2x\sqrt{(1-x^2)}\}$,

(ii) $2\cos^{-1}x=\cos^{-1}(2x^2-1)$,

and (iii) $2\tan^{-1}x=\cos^{-1}\dfrac{1-x^2}{1+x^2}=\sin^{-1}\dfrac{2x}{1+x^2}$.

2. Show that

(i) $3\cos^{-1}x=\cos^{-1}(4x^3-3x)$,

and (ii) $3\tan^{-1}x=\tan^{-1}\dfrac{3x-x^3}{1-3x^2}$. [*Utkal*, 1950]

3. Show that

(i) $\sin^{-1}x\pm\sin^{-1}y=\sin^{-1}\{x(1-y^2)^{1/2}\pm y(1-x^2)^{1/2}\}$,

(ii) $\cos^{-1}x\pm\cos^{-1}y=\cos^{-1}\{xy\mp(1-x^2)^{1/2}(1-y^2)^{1/2}\}$,

(iii) $\cot^{-1}x\pm\cot^{-1}y=\cot^{-1}\dfrac{xy\mp 1}{y\pm x}$.

4. Prove that

$$\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\frac{x+y+z-xyz}{1-yz-zx-xy}.$$

5. Show that $\sin^{-1}\frac{77}{85}=\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}$.

6. Show that $\tan^{-1}\frac{3}{4}=2\tan^{-1}\frac{1}{2}$.

7. Show that

$$\tan^{-1}\tfrac{1}{3}+\tan^{-1}\tfrac{1}{5}+\tan^{-1}\tfrac{1}{7}+\tan^{-1}\tfrac{1}{8}=\tfrac{1}{4}\pi.$$

8. Prove that

$$4\tan^{-1}\tfrac{1}{5}-\tan^{-1}\tfrac{1}{70}+\tan^{-1}\tfrac{1}{99}=\tfrac{1}{4}\pi.$$ [*Luck.*, '64, '67]

9. Solve the equation

$$\tan^{-1}\frac{x-1}{x-2}+\tan^{-1}\frac{x+1}{x+2}=\tfrac{1}{4}\pi.$$ [*Calcutta*, 1947]

10. Solve for x :—

$$\tan^{-1}\tfrac{1}{4}+2\tan^{-1}\tfrac{1}{5}+\tan^{-1}\tfrac{1}{6}+\tan^{-1}1/x=\tfrac{1}{4}\pi.$$

[*Agra*, 1942; *Cal.*, 1949]

11. Solve for x :

$$3\tan^{-1}\frac{1}{2+\sqrt{3}}-\tan^{-1}\frac{1}{x}=\tan^{-1}(1/3).$$

[*Agra*, 1943]

12. Find the value of

$$\sin(\sin^{-1}\tfrac{1}{2}+\cos^{-1}\tfrac{1}{2}).$$

13. Find the tangent of

$$3\tan^{-1}\tfrac{1}{7}+\tan^{-1}\tfrac{1}{3}+\tan^{-1}\tfrac{1}{26}-\tfrac{1}{4}\pi.$$

14. Show that

$$\tan(2\tan^{-1}a)=2\tan(\tan^{-1}a+\tan^{-1}a^3).$$ [*Cal.*, 1944]

15. If $\sec\theta-\operatorname{cosec}\theta=\frac{4}{3}$, show that

$$\theta=\tfrac{1}{2}\sin^{-1}\tfrac{3}{4}.$$

Solve the following equations :

16. $\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}3x.$

[*Gauhati*, '51, '53]

17. $\sin^{-1}\dfrac{2a}{1+a^2}-\cos^{-1}\dfrac{1-b^2}{1+b^2}=\tan^{-1}\dfrac{2x}{1-x^2}.$

18. $\sin[2\cos^{-1}\{\cot(2\tan^{-1}x)\}]=0.$ [*Agra*, 1958]

19. $\frac{1}{2}\cot^{-1}\left(\frac{1-x^2}{2x}\right)+\frac{1}{3}\cot^{-1}\left(\frac{1-3x^2}{3x-x^3}\right)$

$=\pi+2\tan^{-1}(1-x-x^2)$. [*Cal.*, 1944]

20. If $\cos^{-1}(x/a)+\cos^{-1}(y/b)=\alpha$, prove that

$$\frac{x^2}{a^2}-\frac{2xy}{ab}\cos\alpha+\frac{y^2}{b^2}=\sin^2\alpha.$$ [*Cal.*, 1943]

21. Solve the equations

$$a\sin^{-1}x+b\cos^{-1}y=\alpha$$
$$a\cos^{-1}x-b\sin^{-1}y=\beta.$$ [*Cal.*, 1951]

22. If $\cos^{-1}x+\cos^{-1}y+\cos^{-1}z=\pi$, prove that

$$x^2+y^2+z^2+2xyz=1.$$ [*Travancore*, 1959; *Jammu & Kashmir*, 1953]

23. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$, show that

$$x+y+z=xyz.$$ [*Cal.*, 1940]

24. If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\pi$, show that

$$x\sqrt{(1-x^2)}+y\sqrt{(1-y^2)}+z\sqrt{(1-z^2)}=2xyz.$$

[*Delhi*, 1946]

[Solution. Put $\sin^{-1}x=A$, $\sin^{-1}y=B$, and $\sin^{-1}x=C$. Then $A+B+C=\pi$. But when $A+B+C=\pi$, we have $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$, or $\sin A\cos A+\sin B\cos B+\sin C\cos C=2\sin A\sin B\sin C$. Hence on substituting the values we have the desired result.]

25. Solve the equation

$$\theta=\tan^{-1}(2\tan^2\theta)-\tfrac{1}{2}\sin^{-1}\frac{3\sin 2\theta}{5+4\cos 2\theta}.$$

[Hint. $\sin^{-1}\{3\sin 2\theta/(5+4\cos 2\theta)\}=2\tan^{-1}(\tan\theta/3)$, or $2\tan^{-1}(3\cot\theta)$. The first value gives $\tan\theta=0, 1$, or -2; the second value gives only one real value of $\tan\theta$, viz. -1. Hence $\tan\theta=0, \pm 1$, or -2.]

26. Find all the positive integral solutions of

$$\tan^{-1}x+\cot^{-1}y=\tan^{-1}3.$$

27. Write down the general value of

$$\sin^{-1}(-1)^m\tfrac{1}{2},$$

where m is an integer.

28. Show that

$$2b/a = \tan\{\tfrac{1}{4}\pi + \tfrac{1}{2}\cos^{-1}(a/b)\} + \tan\{\tfrac{1}{4}\pi - \tfrac{1}{2}\cos^{-1}(a/b)\}.$$

[*Calcutta*, 1948]

[Hint. Put $\cos^{-1}(a/b) = \theta$.]

29. Prove that

$$\cos[\tan^{-1}\{\sin(\cot^{-1}x)\}] = \left(\frac{x^2+1}{x^2+2}\right)^{1/2}.$$

30. Prove that

$$\tan^{-1}x = 2\tan^{-1}[\operatorname{cosec}(\tan^{-1}x) - \tan(\cot^{-1}x)].$$

31. Show that

$$\tan^{-1}\frac{x}{y} = \tan^{-1}\frac{c_1x-y}{c_1y+x} + \tan^{-1}\frac{c_2-c_1}{c_2c_1+1}$$

$$+\tan^{-1}\frac{c_3-c_2}{c_3c_2+1} + \dots + \tan^{-1}\frac{c_n-c_{n-1}}{c_nc_{n-1}+1} + \tan^{-1}\frac{1}{c_n},$$

where $c_1, c_2, \dots, c_n$ are any quantities whatever.

CHAPTER II

COMPLEX NUMBERS

2·1. Complex numbers. A number of the form $x+iy$, where x and y are real numbers and $i=\sqrt{(-1)}$, is called a *complex number*. A complex number is often called an imaginary number, and a number of the form iy is called a wholly imaginary number.

The necessity of complex numbers is most simply shown by considering the general quadratic equation $ax^2+bx+c=0$, where $b^2<4ac$. Thus if $a=1$, $b=2$ and $c=11$, the roots of the quadratic equation are

$$x=-1\pm\sqrt{(-10)},$$

which remain meaningless as long as complex numbers are not introduced. We therefore introduce a number i such that $i^2=-1$. The number i is assumed to obey all the laws of algebra. We can now write the roots of the quadratic equation $x^2+2x+11=0$ in the form $-1\pm i\sqrt{10}$.

If $z=x+iy$, x is called the *real part* of z and written as $R(z)$, and y the *imaginary part* of z and written as $I(z)$.

Two complex numbers z_1 $(\equiv x_1+iy_1)$ and z_2 $(\equiv x_2+iy_2)$ are said to be equal when

$$R(z_1)=R(z_2) \text{ and } I(z_1)=I(z_2).$$

In other words, if $x_1+iy_1=x_2+iy_2$, then

$$x_1=x_2 \text{ and } y_1=y_2.$$

Complex numbers are assumed to obey the following laws of Algebra :—

1. Addition :

$$(x_1+iy_1)+(x_2+iy_2)=(x_1+x_2)+i(y_1+y_2).$$

2. Subtraction :

$$(x_1+iy_1)-(x_2+iy_2)=(x_1-x_2)+i(y_1-y_2).$$

3. Multiplication :

$$(x_1+iy_1)(x_2+iy_2)=(x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1).$$

4. Division :

$$\frac{(x_1+iy_1)}{(x_2+iy_2)}=\frac{(x_1x_2+y_1y_2)+i(x_2y_1-x_1y_2)}{x_2^2+y_2^2},$$

where in the last case x_2 and y_2 must not be simultaneously zero.

The result of division is based on the fact that if

$$(x_1+iy_1)/(x_2+iy_2)=x_3+iy_3,$$

then we must have

$$(x_1+iy_1)=(x_2+iy_2)(x_3+iy_3).$$

Multiplying both sides by x_2-iy_2, we get

$$(x_1+iy_1)(x_2-iy_2)=(x_2^2+y_2^2)(x_3+iy_3).$$

Using the definition of equality of two complex numbers which amounts to equating the real and imaginary parts on either side we have the values of x_3 and y_3 as stated above.

Of the two numbers $x+iy$ and $x-iy$ one is said to be the *conjugate* of the other. It is obvious that the modulus (see below) of either is the same and further that

$$(x+iy)(x-iy)=x^2+y^2.$$

The last property is useful in putting an expression of the form $1/(a+ib)$ in the form $x+iy$. We have only to multiply the numerator and the deno-

minator of the fraction by the conjugate of the denominator. Thus

$$\frac{1}{a+ib}=\frac{1}{a+ib}\cdot\frac{a-ib}{a-ib}=\frac{a-ib}{a^2+b^2}$$

$$=\frac{a}{a^2+b^2}-i\,\frac{b}{a^2+b^2}.$$

Ex. Express $\dfrac{(2-3i)\ (4+7i)}{(5-6i)\ (-1+9i)}$ in the form $x+iy$.

Multiplying the factors of the numerator and denominator separately, we have

$$\frac{(2-3i)(4+7i)}{(5+6i)\ (-1+9i)}=\frac{29+2i}{-59+39i}=\frac{(29+2i)(-59-39i)}{(-59+39i)(-59-39i)}$$

$$=\frac{-1633-1249i}{5002}=-\frac{1633}{5002}-\frac{1249}{5002}i.$$

2·2. Reduction of complex numbers to the standard form. A complex number $z\equiv x+iy$ can always be put in the form $r(\cos\theta+i\sin\theta)$. For, if

$$x+iy=r(\cos\theta+i\sin\theta),$$

then, on equating the real and imaginary parts, we have

$$x=r\cos\theta \tag{1}$$

and

$$y=r\sin\theta. \tag{2}$$

Squaring and adding (1) and (2), we get

$$r=\sqrt{(x^2+y^2)}. \tag{3}$$

By convention r is taken to be the positive square root of x^2+y^2.

Now from (1) and (2) again, we have

$$\cos\theta=\frac{x}{\sqrt{(x^2+y^2)}}\ \text{and}\ \sin\theta=\frac{y}{\sqrt{(x^2+y^2)}}. \tag{4}$$

There is always one and only one value of θ which satisfies these equations and is such that

$$-\pi < \theta \leqslant \pi.$$

Thus r and θ are determined.

Definition. The number $r[=+\sqrt{(x^2+y^2)}]$ is called the *modulus* of the complex number $z \equiv x+iy$ and is written as

$$r = \text{mod } z, \text{ or } r = |z|$$

[read 'r equal to mod z']; and the angle θ is called the *argument* (or *amplitude*) of the complex number z, and is written as

$$\theta = \arg z \text{ (or amp } z).$$

It is evident that the argument θ has an infinite number of values. That value of θ which lies in the range $-\pi < \theta \leqslant \pi$ and satisfies the equation (4) is called the *principal value* of the argument. When we speak of the argument, we mean, unless otherwise specified, the principal value of the argument.

The student will note that although θ is equal to $\tan^{-1} y/x$ the principal value of the argument is not necessarily the principal value of $\tan^{-1}\left(\frac{y}{x}\right)$ as defined in the previous chapter. For example, the principal value of the argument in the case of the complex number $-1+i$ is $\frac{3}{4}\pi$, but the principal value of $\tan^{-1}(-1)$ is $-\frac{1}{4}\pi$.

If θ be the principal value of the argument then the general value of the argument is denoted by $\theta+2k\pi$, where k is any integer, positive or negative.

The standard form $r(\cos\theta+i\sin\theta)$ is sometimes called the *modulus-amplitude form*.

The following particular cases of such reduction deserve special mention :

(i) $1=\cos 0°+i\sin 0°$, (ii) $-1=\cos\pi+i\sin\pi$,
(iii) $i=\cos\frac{1}{2}\pi+i\sin\frac{1}{2}\pi$, (iv) $-i=\cos(-\frac{1}{2}\pi)+i\sin(-\frac{1}{2}\pi)$.

It may be noted that some writers take the principal value of the argument to be that angle which lies in the range

$$0\leqslant\theta<2\pi.$$

Ex. 1. Express $-\sqrt{3}+i$ in the modulus-amplitude form.

Let $-\sqrt{3}+i$ be equal to $r(\cos\theta+i\sin\theta)$. Equating the real and imaginary parts, we have

$$r\cos\theta=-\sqrt{3}, \qquad \ldots \quad (1)$$

$$r\sin\theta=1. \qquad \ldots \quad (2)$$

Squaring and adding (1) and (2) we get

$$r^2=(-\sqrt{3})^2+1^2=4,$$

giving $$r=2.$$

Substituting this value of r in (1) and (2), we get

$$\cos\theta=-\tfrac{1}{2}\sqrt{3}$$

and $$\sin\theta=\tfrac{1}{2},$$

giving $$\theta=\tfrac{5}{6}\pi.$$

Hence $$-\sqrt{3}+i=2(\cos\tfrac{5}{6}\pi+i\sin\tfrac{5}{6}\pi.)$$

Ex. 2. Put $1+\sqrt{(-3)}$ in the standard form.

Let $1+\sqrt{(-3)}$ be equal to $r(\cos\theta+i\sin\theta)$. Then, equating real and imaginary parts,

$$1=r\cos\theta \text{ and } \sqrt{3}=r\sin\theta.$$

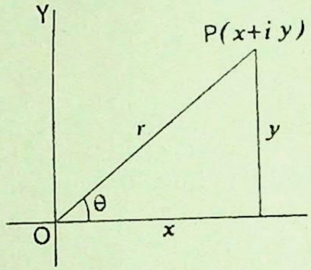
Therefore $r=\sqrt{\{1^2+(\sqrt{3})^2\}}=2$;

and $\cos\theta=\frac{1}{2}$, $\sin\theta=\frac{1}{2}\sqrt{3}$, giving $\theta=\pi/3$.

Hence $1+\sqrt{(-3)}=1+i\sqrt{3}=2\{\cos(\pi/3)+i\sin(\pi/3)\}$.

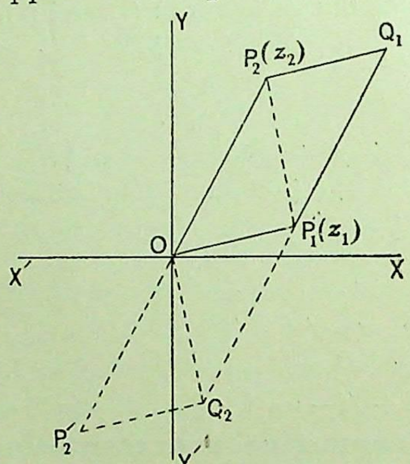
2·3. Geometrical representation of complex numbers. A complex number $z=x+iy$ can be represented by a point P whose coordinates are (x, y) referred to a system of rectangular axes OX and OY. The diagram in which the complex numbers are thus represented by points is known

as the *Argand Diagram**. The axis of x is called the *real axis* and the axis of y the *imaginary axis*. The length OP is the modulus of z and the angle that OP makes with the positive direction of the x-axis is the principal value of the argument.



By the *affix* of the point P is meant the complex number z.

2·4. Geometrical representation of $z_1 \pm z_2$. To represent the sum or difference of two complex numbers suppose that P_1 and P_2 are the points



corresponding to the complex numbers $z_1(\equiv x_1+iy_1)$ and $z_2(\equiv x_2+iy_2)$.

*Called after the name of Argand, who first gave this method in a tract published in 1806 A.D.

If we complete the parallelogram $OP_1Q_1P_2$, the point corresponding to z_1+z_2 would be Q_1, since the coordinates of Q_1 are evidently (x_1+x_2, y_1+y_2).

Since $$OQ_1 \leqslant OP_1 + P_1Q_1$$

we get $$|z_1+z_2| \leqslant |z_1| + |z_2|,$$

that is, *the modulus of the sum of two complex numbers is less than or equal to the sum of their moduli.* This result is true also for more than two complex numbers.

Again if we produce P_2O to P'_2, taking $OP'_2 = OP_2$, P'_2 will be the point $(-x_2, -y_2)$, and so it corresponds to the complex number $-x_2-iy_2$; consequently, the point corresponding to z_1-z_2 would be Q_2, where $OP_1Q_2P'_2$ is a parallelogram.

Since OP_2 is equal and parallel to Q_2P_1, therefore $OQ_2P_1P_2$ is a parallelogram. Hence it follows that

(1) *the modulus of z_1-z_2 is equal to the distance between the points z_2 and z_1*; and

(2) *the argument of z_1-z_2 is the angle which the line directed from the point z_2 to the point z_1 makes with the positive direction of the real axis.*

Also since $$P_1P_2 \geqslant OP_1 \sim OP_2,$$

we get $$|z_1-z_2| \geqslant |z_1| \sim |z_2|,$$

that is, *the modulus of the difference of two complex numbers is greater than or equal to the difference of their moduli.*

2·5. Product of any number of complex numbers. Consider the case of two complex numbers $z_1(\equiv x_1+iy_1)$ and $z_2(\equiv x_2+iy_2)$.

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If $|z_1|=r_1$, $|z_2|=r_2$ and $\arg(z_1)=\theta_1$, $\arg(z_2)=\theta_2$, we have

$$z_1=x_1+iy_1=r_1(\cos\theta_1+i\sin\theta_1),$$

and $$z_2=x_2+iy_2=r_2(\cos\theta_2+i\sin\theta_2).$$

Hence

$$\begin{aligned}z_1z_2&=r_1r_2(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)\\&=r_1r_2\{\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\}.\end{aligned}$$

It follows that

$$|z_1z_2|=|z_1|\times|z_2|,$$

and $$\arg(z_1z_2)=\arg(z_1)+\arg(z_2),$$

where we have taken the principal values of the arguments concerned. These results are true for any number of complex numbers, and we may put the conclusions as follows:

(1) *The modulus of the product of any number of complex numbers is equal to the product of their moduli.*

(2) *The argument of the product of any number of complex numbers is equal to the sum of their arguments.*

2·6. Division of one complex number by another. Adopting the notation of the above article, we have

$$\begin{aligned}\frac{z_1}{z_2}&=\frac{r_1(\cos\theta_1+i\sin\theta_1)}{r_2(\cos\theta_2+i\sin\theta_2)}\\&=(r_1/r_2)\{\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2).\end{aligned}$$

Hence

$$\frac{z_1}{z_2}=\frac{r_1}{r_2},\quad\text{and}\quad\arg\left(\frac{z_1}{z_2}\right)=\theta_1-\theta_2.$$

It follows that

(*i*) *The modulus of the quotient of two complex numbers is equal to the quotient of their moduli; and*

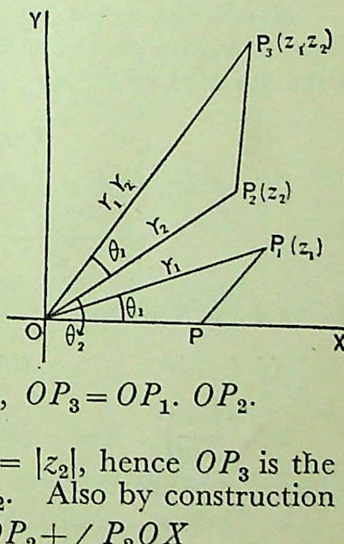
(*ii*) *the argument of the quotient of two complex numbers is equal to the difference of their arguments.*

2·7. Geometrical representation of z_1z_2 and z_2/z_1. Let the affixes of the points P_1 and P_2 be z_1 and z_2, where

$$z_1 = x_1 + iy_1 = r_1(\cos\theta_1 + i_1 \sin\theta_1),$$

and $$z_2 = x_2 + iy_2 = r_2(\cos\theta_2 + i \sin\theta_2).$$

Consider a point P on the real axis, such that $OP = 1$, i.e., the affix of the point P is $1 + 0i$.



Draw a triangle OP_2P_3 similar to the triangle OPP_1 on the side of P_2 remote from P_1, such that $\angle P_2OP_3 = \theta_1$. Then from a property of similar triangles,

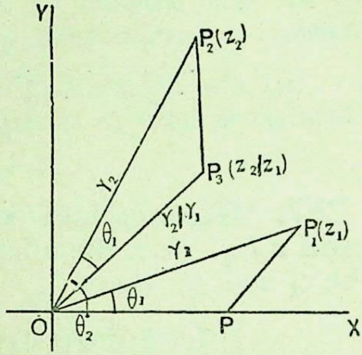
$$OP_2/OP_3 = OP/OP_1, \text{ i.e., } OP_3 = OP_1 . OP_2.$$

But $OP_1 = |z_1|$ and $OP_2 = |z_2|$, hence OP_3 is the modulus of the product z_1z_2. Also by construction

$$\angle P_3OX = \angle P_3OP_2 + \angle P_2OX$$
$$= \theta_1 + \theta_2.$$

Hence P_3 represents the complex number z_1z_2.

If, however, we draw the triangle OP_2P_3 similar to the triangle OP_1P on the same side of P_2 as P_1, as shown in the adjoining figure, we have



$$\frac{OP_3}{OP_2}=\frac{OP}{OP_1},$$

i.e. $$OP_3=\frac{OP_2}{OP_1}.$$

Also $\angle P_3OX=\angle P_2OX$
$-\angle P_2OP_3$
$=\theta_2-\theta_1.$

Hence the point P_3 is such that $OP_3=r_2/r_1$ and its inclination to the real axis is $\theta_2-\theta_1$. It follows that the affix of P_3 is z_2/z_1.

Examples

1. $x=\cos\alpha+i\sin\alpha$, $y=\cos\beta+i\sin\beta$, show that

$$\frac{(x+y)(xy-1)}{(x-y)(xy+1)}=\frac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta}.$$

2. Express the following numbers in the form $A+iB$ where A and B are real numbers :

(i) $(1+i)^2$, (ii) $\dfrac{1}{1+i}$, (iii) $\dfrac{1+i}{1-i}$,

(iv) $\left(\dfrac{1-i}{1+i}\right)^2$, (v) $\left(\dfrac{a+ib}{a-ib}\right)^2-\left(\dfrac{a-ib}{a+ib}\right)^2$,

where a and b are real.

3. Find the modulus and argument of the following :

(i) $\frac{1}{2}+i\frac{1}{2}\sqrt{3}$, (ii) $1+\cos\theta+i\sin\theta$,

(iii) $1-\cos\theta+i\sin\theta$, (iv) $1-i$,

(v) $-5-12i$, (vi) $2-3i$.

4. A variable complex number $z=x+iy$ is such that the amplitude of the fraction $\dfrac{z-1}{z+1}$ is always equal to $\pi/4$. Show that

$$x^2+y^2-2y=1.$$

5. Let (r, θ) denote the complex number whose modulus is r and amplitude θ. Then if $a\equiv(1, \alpha)$, $b\equiv(1, \beta)$, $c\equiv(1, \gamma)$, and if $a+b+c=0$, prove that

$$a^{-1}+b^{-1}+c^{-1}=0.$$ [*Punjab*, 1945]

6. If $x+iy=3/(2+\cos\theta+i\sin\theta)$, prove that

$$(x-1)(x-3)+y^2=0.$$ [*Madras*, 1940]

7. If $l^2+m^2+n^2=1$, and $(m+in)=(1+l)z$, show that

$$\frac{l+im}{1+n}=\frac{1+iz}{1-iz}.$$ [*Nagpur*, 1948]

8. Show that the representative points of the complex numbers

$$i, -2-5i, \quad 1+4i \text{ and } 3+10i$$

are collinear.

9. The vertices of a triangle are represented in an Argand diagram by the complex numbers z_1, z_2, z_3. Interpret the modulus and argument of $(z_2-z_1)/(z_3-z_1)$ in terms of the sides and angles of the triangle.

10. Show that the two lines joining the points $z=a$, $z=b$ and $z=c$, $z=d$ will be perpendicular if

$$\arg\frac{a-b}{c-d}=\pm\tfrac{1}{2}\pi,$$

i.e., if $(a-b)/(c-d)$ is purely imaginary.

11. If z_1 and z_2 are two non-zero complex numbers, prove that

$$|z_1+z_2|^2+|z_1-z_2|^2=2|z_1|^2+2|z_2|^2.$$

Prove also that

$$|z_1+z_2|^2=|z_1|^2+|z_2|^2,$$

if and only if z_1/z_2 is purely imaginary.

12. Prove that the two triangles whose vertices in the Argand diagram have affixes a, b, c and α, β, γ will be similar if

$$(b\gamma - c\beta) + (c\alpha - a\gamma) + (a\beta - b\alpha) = 0.$$

[Hint. The two triangles will be similar if

$$\frac{b-c}{\beta-\gamma} = \frac{c-a}{\gamma-\alpha} = \frac{a-b}{\alpha-\beta}.\ ]$$

CHAPTER III

DE MOIVRE'S THEOREM

3·1. De Moivre's* Theorem. *Whatever be the value of n, integral or fractional, positive or negative, the value or one of the values of*

$$(\cos\theta+i\sin\theta)^n$$

is

$$\cos n\theta+i\sin n\theta.$$

CASE I. Let n be a positive integer.

By actual multiplication, we have

$$(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)$$
$$=\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2 +i(\cos\theta_1\sin\theta_2+\sin\theta_1\cos\theta_2)$$
$$=\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2).$$

Similarly,

$$(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)(\cos\theta_3+i\sin\theta_3)$$
$$=\{\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2)\}(\cos\theta_3+i\sin\theta_3)$$
$$=\cos(\theta_1+\theta_2+\theta_3)+i\sin(\theta_1+\theta_2+\theta_3).$$

Proceeding in this way to n factors of the form $\cos\alpha+i\sin\alpha$, we obtain

$$(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)\ldots(\cos\theta_n+i\sin\theta_n)$$
$$=\cos(\theta_1+\theta_2+\ldots+\theta_n)+i\sin(\theta_1+\theta_2+\ldots+\theta_n).$$

Putting in this

$$\theta_1=\theta_2=\ldots=\theta_n=\theta,$$

we have

$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta,$$

*Called after the name of the French mathematician Abraham de Moivre (1667-1754), who discovered this theorem.

which is De Moivre's theorem when n is a positive integer.

[When n is a positive integer, the theorem may also be proved by mathematical induction, as follows:

Suppose that the theorem is true for a particular value of n, i.e., let $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin\theta$. Then, multiplying both sides by $\cos\theta+i\sin\theta$, we have

$$\begin{aligned}(\cos\theta+i\sin\theta)^{n+1}&=(\cos n\theta+i\sin n\theta)(\cos\theta+i\sin\theta)\\&=(\cos n\theta\cos\theta-\sin n\theta\sin\theta)\\&\quad+i(\sin n\theta\cos\theta+\cos n\theta\sin\theta)\\&=\cos(n+1)\theta+i\sin(n+1)\theta.\end{aligned}$$

Hence if the theorem is true for any integral value of n, it is also true for the next integral value of n.

But it is easy to see that the theorem is true for $n=2$; for

$$\begin{aligned}(\cos\theta+i\sin\theta)^2&=(\cos\theta+i\sin\theta)(\cos\theta+i\sin\theta)\\&=(\cos^2\theta-\sin^2\theta)+2i\sin\theta\cos\theta\\&=\cos 2\theta+i\sin 2\theta.\end{aligned}$$

Hence the theorem is true for $n=3$, and therefore for $n=4$, $n=5$, etc. Hence the theorem is true for all positive integral values of n.]

Case II. Let n be a negative integer.

Suppose that $n=-m$, where m is a positive integer. Then

$$\begin{aligned}&(\cos\theta+i\sin\theta)^n\\&\quad=(\cos\theta+i\sin\theta)^{-m},\\&\quad=\frac{1}{(\cos\theta+i\sin\theta)^m},\\&\quad=\frac{1}{(\cos m\theta+i\sin m\theta)},\text{ by Case I above}\\&\quad=\frac{\cos m\theta-i\sin m\theta}{(\cos m\theta+i\sin m\theta)(\cos m\theta-i\sin m\theta)},\end{aligned}$$

on multiplying the numerator and denominator by $\cos m\theta - i \sin m\theta$,

$$= \frac{\cos m\theta - i \sin m\theta}{\cos^2 m\theta + \sin^2 m\theta},$$

$$= \cos m\theta - i \sin m\theta,$$

$$= \cos(-n\theta) - i \sin(-n\theta), \text{ writing } m = -n$$

$$= \cos n\theta + i \sin n\theta.$$

This proves De Moivre's theorem when n is a negative integer.

CASES I and II together give us that when n is an integer, positive or negative,

$$(\cos\theta + i \sin\theta)^n = \cos n\theta + i \sin n\theta.$$

It follows that $\cos\theta + i \sin\theta$ is one of the values of $(\cos n\theta + i \sin n\theta)^{1/n}$, when n is any integer.

CASE III. Let n be a fraction, positive or negative.

Suppose that $n = p/q$, where p is a positive or negative integer and q a positive integer. Then

$$(\cos\theta + i \sin\theta)^n = (\cos\theta + i \sin\theta)^{p/q}$$

$$= (\cos p\theta + i \sin p\theta)^{1/q}$$

and by what has just been shown, one of the values of the last expression is

$$\cos\frac{p}{q}\theta + i \sin\frac{p}{q}\theta.$$

Hence $\cos n\theta + i \sin n\theta$ is one of the values of $(\cos\theta + i \sin\theta)^n$, when n is a fraction.

Thus De Moivre's theorem is completely established.

Some writers use the notation cis θ to denote $\cos\theta+i\sin\theta$. In this notation, De Moivre's theorem would be that cis $n\theta$ is the value or one of the values of $(\text{cis }\theta)^n$, whatever be the value of n, integral or fractional, positive or negative.

Corollary 1. For all values of n, integral or fractional, positive or negative, $\cos n\theta-i\sin n\theta$ is the value or one of the values of $(\cos\theta-i\sin\theta)^n$.

Corollary 2. $\dfrac{1}{\cos\theta\pm i\sin\theta}=\cos\theta\mp i\sin\theta.$

Ex. 1. Prove that

$$\left(\frac{\cos\theta+i\sin\theta}{\sin\theta+i\cos\theta}\right)^4=\cos 8\theta+i\sin 8\theta. \quad [\textit{Andhra}, 1934]$$

By De Moivre's theorem,

$$(\cos\theta+i\sin\theta)^4=\cos 4\theta+i\sin 4\theta.$$

Also

$$\sin\theta+i\cos\theta=\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right).$$

Therefore by De Moivre's theorem

$$\begin{aligned}(\sin\theta+i\cos\theta)^4&=\cos 4\left(\frac{\pi}{2}-\theta\right)+i\sin 4\left(\frac{\pi}{2}-\theta\right)\\&=\cos(2\pi-4\theta)+i\sin(2\pi-4\theta)\\&=\cos 4\theta-i\sin 4\theta.\end{aligned}$$

Hence

$$\begin{aligned}\left(\frac{\cos\theta+i\sin\theta}{\sin\theta+i\cos\theta}\right)^4&=\frac{\cos 4\theta+i\sin 4\theta}{\cos 4\theta-i\sin 4\theta}\\&=(\cos 4\theta+i\sin 4\theta)^2, \text{ by Cor. 2}\\&=\cos 8\theta+i\sin 8\theta.\end{aligned}$$

Ex. 2. Prove that

$$\left[\frac{1+\sin\phi+i\cos\phi}{1+\sin\phi-i\cos\phi}\right]^n=\cos\left(\tfrac{1}{2}n\pi-n\phi\right)+i\sin\left(\tfrac{1}{2}n\pi-n\phi\right).$$

[*Jammu & Kashmir*, 1953]

We first express $1+\sin\phi+i\cos\phi$ and $1+\sin\phi-i\cos\phi$ in the modulus-amplitude form.

Put $1+\sin\phi = r\cos\theta$ and $\cos\phi = r\sin\theta$.

Then

$$\tan\theta = \frac{\cos\phi}{1+\sin\phi} = \frac{\cos^2\frac{1}{2}\phi - \sin^2\frac{1}{2}\phi}{\cos^2\frac{1}{2}\phi + \sin^2\frac{1}{2}\phi + 2\sin\frac{1}{2}\phi\cos\frac{1}{2}\phi}$$

$$= \frac{\cos\frac{1}{2}\phi - \sin\frac{1}{2}\phi}{\cos\frac{1}{2}\phi + \sin\frac{1}{2}\phi} = \frac{1-\tan\frac{1}{2}\phi}{1+\tan\frac{1}{2}\phi}$$

$$= \tan\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\phi\right)$$

or $\qquad \theta = \frac{1}{4}\pi - \frac{1}{2}\phi$,

so that $1+\sin\phi + i\cos\phi = r\{\cos(\frac{1}{4}\pi - \frac{1}{2}\phi) + i\sin(\frac{1}{4}\pi - \frac{1}{2}\phi)\}$

and $\qquad 1+\sin\phi - i\cos\phi = r\{\cos(\frac{1}{4}\pi - \frac{1}{2}\phi) - i\sin(\frac{1}{4}\pi - \frac{1}{2}\phi)\}$.

It follows that

$$\frac{1+\sin\phi + i\cos\phi}{1+\sin\phi - i\cos\phi} = \frac{\cos(\frac{1}{4}\pi - \frac{1}{2}\phi) + i\sin(\frac{1}{4}\pi - \frac{1}{2}\phi)}{\cos(\frac{1}{4}\pi - \frac{1}{2}\phi) - i\sin(\frac{1}{4}\pi - \frac{1}{2}\phi)}$$

$$= \cos(\tfrac{1}{2}\pi - \phi) + i\sin(\tfrac{1}{2}\pi - \phi), \text{ by Cor. 2.}$$

Hence by De Moivre's theorem

$$\left[\frac{1+\sin\phi + i\cos\phi}{1+\sin\phi - i\cos\phi}\right]^n = \{\cos(\tfrac{1}{2}\pi - \phi) + i\sin(\tfrac{1}{2}\pi - \phi)\}^n$$

$$= \cos(\tfrac{1}{2}n\pi - n\phi) + i\sin(\tfrac{1}{2}n\pi - n\phi).$$

Ex. 3. If $2\cos\theta = x + \dfrac{1}{x}$ and $2\cos\phi = y + \dfrac{1}{y}$, show that one of the values of $x^m/y^n + y^n/x^m$ is $2\cos(m\theta - n\phi)$.

[*Lucknow*, 1968]

Since $\qquad 2\cos\theta = x + \dfrac{1}{x}$,

therefore $\qquad x^2 - 2x\cos\theta + 1 = 0$,

or $\qquad x = \cos\theta \pm i\sin\theta$.

Hence one of the values of x is $\cos\theta + i\sin\theta$.

Similarly, one of the values of y is $\cos\phi + i\sin\phi$.

Consequently, one of the values of $x^m/y^n + y^n/x^m$ is

$$\frac{(\cos\theta + i\sin\theta)^m}{(\cos\phi + i\sin\phi)^n} + \frac{(\cos\phi + i\sin\phi)^n}{(\cos\theta + i\sin\theta)^m}$$

or

$$\frac{\cos m\theta + i\sin m\theta}{\cos n\phi + i\sin n\phi} + \frac{\cos n\phi + i\sin n\phi}{\cos m\theta + i\sin m\theta}$$

or $$\{\cos(m\theta-n\phi)+i\sin(m\theta-n\phi)\}$$
$$+\{\cos(m\theta-n\phi)-i\sin(m\theta-n\phi)\}$$

or $$2\cos(m\theta-n\phi).$$

Ex. 4. If $\sin\alpha+\sin\beta+\sin\gamma=\cos\alpha+\cos\beta+\cos\gamma=0$, show that

$$\cos 3\alpha+\cos 3\beta+\cos 3\gamma=3\cos(\alpha+\beta+\gamma),$$

and $$\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma).$$

[*Lucknow*, 66; *Poona*, '58]

We know that if $a+b+c=0$, then

$$a^3+b^3+c^3=3abc.$$

Now if we assume that

$$a=\cos\alpha+i\sin\alpha,$$
$$b=\cos\beta+i\sin\beta,$$

and $$c=\cos\gamma+i\sin\gamma,$$

then $a+b+c=0$; therefore

$$(\cos\alpha+i\sin\alpha)^3+(\cos\beta+i\sin\beta)^3+(\cos\gamma+i\sin\gamma)^3$$
$$=3(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)(\cos\gamma+i\sin\gamma),$$

or $$(\cos 3\alpha+i\sin 3\alpha)+(\cos 3\beta+i\sin 3\beta)+(\cos 3\gamma+i\sin 3\gamma)$$
$$=3\{\cos(\alpha+\beta+\gamma)+i\sin(\alpha+\beta+\gamma)\}.$$

Hence, equating the real and imaginary parts on the two sides, we have the required results.

EXAMPLES

Simplify :

1. $(\cos\theta+i\sin\theta)^5(\cos\theta-i\sin\theta)^3$.

2. $\dfrac{(\cos\theta+i\sin\theta)^4}{(\cos\theta-i\sin\theta)^3}$.

3. $\dfrac{(\cos 3\theta+i\sin 3\theta)^5(\cos\theta-i\sin\theta)^3}{(\cos 5\theta+i\sin 5\theta)^7(\cos 2\theta-i\sin 2\theta)^5}$. [*Lucknow*, '67]

4. Express $(1+7i)/(2-i)^2$ in the forms

(i) $x+iy$, (ii) $r(\cos\theta+i\sin\theta)$,

and deduce or prove otherwise that its fourth power is a real negative number.

5. Show that

$$\frac{(\cos 2\theta + i \sin 2\theta)^3 (\cos 3\theta - i \sin 3\theta)^4}{(\cos 3\theta + i \sin 3\theta)^2 (\cos 4\theta + i \sin 4\theta)^{-3}} = 1.$$

6. If $a = \cos 2\alpha + i \sin 2\alpha$, $b = \cos 2\beta + i \sin 2\beta$, $c = \cos 2\gamma + i \sin 2\gamma$, and $d = \cos 2\delta + i \sin 2\delta$, prove that

(i) $a + b = 2 \cos (\alpha - \beta)\{\cos (\alpha + \beta) + i \sin (\alpha + \beta)\}$,

(ii) $a - b = 2i \sin (\alpha - \beta)\{\cos (\alpha + \beta) + i \sin (\alpha + \beta)\}$,

(iii) $ab + cd = 2 \cos (\alpha + \beta - \gamma - \delta)\{\cos (\alpha + \beta + \gamma + \delta) + i \sin (\alpha + \beta + \gamma + \delta)\}$,

and (iv) $(a + b)(c + d) = 4 \cos (\alpha - \beta) \cos (\gamma - \delta) \times \{\cos (\alpha + \beta + \gamma + \delta) + i \sin(\alpha + \beta + \gamma + \delta)\}$.

7. If $z^2 - 2z \cos \theta + 1 = 0$, show that $z^2 + z^{-2} = 2 \cos 2\theta$ and $z^3 - z^{-3} = 2i \sin 3\theta$.

8. Use De Moivre's theorem to form an equation whose roots are the nth powers of the roots of the equation

$$x^2 - 2x \cos \theta + 1 = 0.$$

9. If $x_r = \cos (\pi/2^r) + i \sin(\pi/2^r)$, prove that

$$x_1 x_2 x_3 \ldots \infty = -1.$$ [*Poona*, 1957]

10. Prove that

$$(1 + \cos \theta + i \sin \theta)^n + (1 + \cos \theta - i \sin \theta)^n = 2^{n+1} \cos^n \frac{\theta}{2} \cos \frac{n\theta}{2}.$$

[*Madras*, 1936]

11. Prove that

$(1 + i)^n + (1 - i)^n = 2^{n/2+1} \cos \frac{1}{4} n\pi$. [*Agra*, '47; *Andhra*, '52]

12. If $(1 + x)^n = p_0 + p_1 x + p_2 x^2 + \ldots$, show that

$$p_0 - p_2 + p_4 - \ldots = 2^{n/2} \cos \tfrac{1}{4} n\pi,$$

and $$p_1 - p_3 + p_5 - \ldots = 2^{n/2} \sin \tfrac{1}{4} n\pi.$$ [*Luck.*, 1967]

13. If $\sin x + \sin y + \sin z = 0 = \cos x + \cos y + \cos z$, show that

$$\sin 2x + \sin 2y + \sin 2z = 0 = \cos 2x + \cos 2y + \cos 2z.$$

[Hint. Use the following identity : If $a + b + c = 0$, and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$, then $a^2 + b^2 + c^2 = 0$.]

14. If $\sin \alpha + \sin \beta + \sin \gamma = 0 = \cos \alpha + \cos \beta + \cos \gamma$, prove that

$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = \frac{3}{2}$. [*Luck.*, '68].

15. From the identity

$$\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}+\frac{(x-a)(x-b)}{(c-a)(c-b)}=1,$$

deduce, by assuming $x=\cos 2\theta+i\sin 2\theta$ and making corresponding assumptions for a, b, and c, that

$$\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}\sin 2(\theta-\alpha)+\text{two similar expressions}=0.$$

3·2. The q roots of $(\cos\theta+i\sin\theta)^{p/q}$, p and q being integers prime to each other. We have seen in § 3·1 that when n is fractional $\cos n\theta+i\sin n\theta$ is one of the values of $(\cos\theta+i\sin\theta)^n$. We shall now obtain all the values of the last expression.

We notice that the expression $\cos\theta+i\sin\theta$ is not altered if for θ we put $\theta+2r\pi$, where r is any integer. That is,

$$(\cos\theta+i\sin\theta)^n=\{\cos(\theta+2r\pi)+i\sin(\theta+2r\pi)\}^n,$$

where $r=0, 1, 2, 3, \ldots$.

If $n=p/q$, where p and q are integers, one of the values of the expression on the right of the above equality is

$$\cos\left\{\frac{p}{q}(\theta+2r\pi)\right\}+i\sin\left\{\frac{p}{q}(\theta+2r\pi)\right\}. \tag{1}$$

Giving to r the values $0, 1, 2, \ldots, q-1$, we obtain the set

$$\cos\left(\frac{p}{q}\theta\right)+i\sin\left(\frac{p}{q}\theta\right),$$

$$\cos\left\{\frac{p}{q}(\theta+2\pi)\right\}+i\sin\left\{\frac{p}{q}(\theta+2\pi)\right\},$$

$$\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots$$

$$\cos\left[\frac{p}{q}\{\theta+2(q-2)\pi\}\right]+i\sin\left[\frac{p}{q}\{\theta+2(q-2)\pi\}\right],$$

and $\cos\left[\frac{p}{q}\{\theta+2(q-1)\pi\}\right]+i\sin\left[\frac{p}{q}\{\theta+2(q-1)\pi\}\right]$.

If in (1) we put $r=q$, $q+1$, ..., this set is repeated again. Moreover, no two quantities of this set have the same value, since no two of the angles involved are either equal or differ by a multiple of 2π. We thus obtain q distinct values of the expression $\cos\{p(\theta+2r\pi)/q\}+i\sin\{p(\theta+2r\pi)/q\}$ and these are the values of $(\cos\theta+i\sin\theta)^{p/q}$.

Thus we see that $(\cos\theta+i\sin\theta)^{p/q}$, where p/q is a rational fraction in its lowest terms, has got q and only q distinct values and these are obtained on successively putting $r=0$, 1, 2, ..., $(q-1)$ in the expression

$$\cos\left\{\frac{p}{q}(\theta+2r\pi)\right\}+i\sin\left\{\frac{p}{q}(\theta+2r\pi)\right\}.$$

Further, since

$$\cos\left\{\frac{p}{q}(\theta+2r\pi)\right\}+i\sin\left\{\frac{p}{q}(\theta+2r\pi)\right\}$$

$$=\{\cos(p\theta/q)+i\sin(p\theta/q)\}\{\cos(2r\pi p/q)+i\sin(2r\pi p/q)\}$$

$$=\{\cos(p\theta/q)+i\sin(p\theta/q)\}\{\cos(2\pi p/q)+i\sin(2\pi p/q)\}^r,$$

therefore, the q roots of $(\cos\theta+i\sin\theta)^{p/q}$ may be arranged in the geometrical progression

$$z,\ wz,\ w^2z,\ \ldots,\ w^{q-1}z,$$

where $z=\cos(p\theta/q)+i\sin(p\theta/q)$,

and $w=\cos(2\pi p/q)+i\sin(2\pi p/q)$.

3·21. The above article is useful in finding the roots of any given quantity. We have only to put the quantity in the form $r(\cos\theta+i\sin\theta)$ and proceed as above.

3·22. The student should note carefully that $(\cos\theta+i\sin\theta)^{2/16}$ has only 8 values and no more and that these eight values are the values of $(\cos\theta+i\sin\theta)^{1/8}$. Therefore, in order to find the different values of $(\cos\theta+i\sin\theta)^{p/q}$, he should see whether p/q is in its lowest terms. If p/q is not reduced to its lowest terms, the values will be repeated.

Ex. 1. Write down the nth roots of -1 and show that (i) no two of these are equal and that (ii) any one root can be expressed as a power of any other.

Suppose that $-1=r(\cos\theta+i\sin\theta)$, then

$$-1=r\cos\theta,\ 0=r\sin\theta,$$

giving $r=1$ and $\theta=\pi$.

Hence

$$(-1)^{1/n}=(\cos\pi+i\sin\pi)^{1/n}$$
$$=[\cos\{(2k\pi+\pi)/n\}+i\sin\{(2k\pi+\pi)/n\}],$$

where k is any positive integer, including zero.

Giving to k the values $0, 1, 2, \ldots, n-1$, we get

$$\cos(\pi/n)+i\sin(\pi/n),\ \cos(3\pi/n)+i\sin(3\pi/n),\ \ldots,$$
$$\cos\{(2n-1)\pi/n\}+i\sin\{(2n-1)\pi/n\}.$$

These values repeat for $k=n$, $n+1$, etc. Hence the above are the nth roots of -1 and they are all different.

Now a root of -1 is

$$\frac{\cos(2r+1)\pi}{n}+i\sin\frac{(2r+1)\pi}{n},\ r \text{ being an integer.}$$

But

$$\cos\frac{(2r+1)\pi}{n}+i\sin\frac{(2r+1)\pi}{n}=\left(\cos\frac{\pi}{n}+i\sin\frac{\pi}{n}\right)^{2r+1}.$$

Since $\cos\pi/n+i\sin\pi/n$ is a root, any one root can be expressed as a power of any other.

Ex. 2. Solve $x^5=1$ by De Moivre's theorem and prove that the sum of the nth powers of the roots of the equation, n being an integer not divisible by 5, is zero. [*Cal.*, 1940]

We have

$$x^5=1$$
$$=\cos 2r\pi+i\sin 2r\pi.$$

$$\therefore\quad x=(\cos 2r\pi+i\sin 2r\pi)^{1/5}$$
$$=\cos(2r\pi/5)+i\sin(2r\pi/5),\ \text{where } r=0, 1, 2, 3, 4.$$

Hence the roots of the above equation are 1, $\cos(2\pi/5)+i\sin(2\pi/5)$, $\cos(4\pi/5)+i\sin(4\pi/5)$, $\cos(6\pi/5)+i\sin(6\pi/5)$, and $\cos(8\pi/5)+i\sin(8\pi/5)$.

The sum of the nth powers of these roots

$$=1^n+\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)^n+\left(\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}\right)^n$$

$$+\left(\cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5}\right)^n+\left(\cos\frac{8\pi}{5}+i\sin\frac{8\pi}{5}\right)^n$$

$$=1+\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)^n+\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)^{2n}$$

$$+\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)^{3n}+\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)^{4n}$$

$$=\frac{1-\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)^{5n}}{1-\left(\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}\right)^n}$$

$$=\frac{1-(\cos 2n\pi+i\sin 2n\pi)}{1-\left(\cos\frac{2n\pi}{5}+i\sin\frac{2n\pi}{5}\right)}$$

$$=0,$$

because, n being not a multiple of 5, $\cos(2n\pi/5)+i\sin(2n\pi/5)\neq 1$.

NOTE. If n be a multiple of 5, the nth power of each root will be equal to 1, so that the sum of the nth powers of all the five roots will be equal to 5.

Ex. 3. Prove by the use of De Moivre's theorem that the roots of the equation $(x-1)^n=x^n$ (n being a positive integer) are $\frac{1}{2}\{1+i\cot(r\pi/n)\}$, where r has the values $0, 1, 2, \ldots, (n-1)$.

Since $\qquad (x-1)^n=x^n,$

therefore $\qquad \left(\frac{x-1}{x}\right)^n=1$

$$=\cos 2r\pi+i\sin 2r\pi,$$

where $r=0, 1, \ldots, (n-1)$.

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Therefore $\dfrac{x-1}{x}=\cos\dfrac{2r\pi}{n}+i\sin\dfrac{2r\pi}{n}$,

$$\text{or}\quad x=\frac{1}{1-\cos\dfrac{2r\pi}{n}-i\sin\dfrac{2r\pi}{n}}$$

$$=\frac{1}{2\sin^2\dfrac{r\pi}{n}-i\,.\,2\sin\dfrac{r\pi}{n}\cos\dfrac{r\pi}{n}}$$

$$=\frac{-1}{2i\sin\dfrac{r\pi}{n}\left(\cos\dfrac{r\pi}{n}+i\sin\dfrac{r\pi}{n}\right)}$$

$$=\frac{i}{2\sin\dfrac{r\pi}{n}\left(\cos\dfrac{r\pi}{n}+i\sin\dfrac{r\pi}{n}\right)}$$

$$=\frac{i\left(\cos\dfrac{r\pi}{n}-i\sin\dfrac{r\pi}{n}\right)}{2\sin\dfrac{r\pi}{n}}$$

$$=\tfrac{1}{2}\left(1+i\cot\frac{r\pi}{n}\right),\ \text{where } r=0,\ 1,\ \ldots,\ (n-1).$$

Examples

1. Find the values of

   (i) $(+1)^{1/3}$, (ii) $(-1)^{1/3}$.

2. Find all the values of

   (i) $(1+i)^{1/7}$, [*Cal.*, 1940]

   (ii) $(1+i)^{2/3}$, [*Agra*, 1943]

   (iii) $(\sqrt{3}+i)^{1/3}$, [*Wales*, 1945]

   (iv) $(i-1)^{1/6}$. [*Poona*, 1945]

3. Express $P\equiv\dfrac{(\sqrt{3}-1)+i(\sqrt{3}+1)}{2\sqrt{2}}$

in the form $r(\cos\theta+i\sin\theta)$ and derive all the four values of $P^{1/4}$. [*Cal.*, 1942]

4. Solve $x^7 = 1$ by the help of De Moivre's theorem. [*Allahabad*, 1950]

5. Solve the equation $x^9 - x^5 + x^4 - 1 = 0$. [*Lucknow*, 1959]

6. Find the continued product of the four values of $(\cos \frac{1}{3}\pi + i \sin \frac{1}{3}\pi)^{3/4}$.

7. Find the cube roots of $8i$, expressing each in the form $a+ib$. [*Liverpool*, 1946]

8. Find the cube roots of $1 - \cos\phi - i\sin\phi$, where ϕ is real, and state the argument and modulus of each.

9. Find the roots of the equation $x^3 + 8 = 0$, and mark them on an Argand diagram.

10. Solve $x^{12} - 1 = 0$, and indicate if there is any relation between these roots and the roots of the equation $x^4 + x^2 + 1 = 0$. [*Lucknow*, 1959]

3·3. Binomial Theorem. The student is already familiar with the Binomial Theorem for real quantities. We shall now state the corresponding theorem for complex quantities.

(1) *If z_1 and z_2 are complex quantities and n is a positive integer, then*

$$(z_1+z_2)^n = z_1^n + nz_1^{n-1} z_2 + \frac{n(n-1)}{2!} z_1^{n-2} z_2^2 + \dots + \frac{n(n-1)}{2!} z_1^2 z_2^{n-2} + nz_1 z_2^{n-1} + z_2^n.$$

This is the Binomial theorem for a positive integral index, and can be easily proved by the ordinary method.

(2) *If z is a complex quantity and n a negative integer or a positive or negative fraction, then*

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{2!} z^2 + \frac{n(n-1)(n-2)}{3!} z^3 + \dots \textit{ad inf.},$$

provided $|z| < 1$.

When $|z|=1$, *this result is still true if* (*i*) $n>0$, *or if* (*ii*) $-1<n<0$ *and* $z\neq -1$.

This is the Binomial theorem for a negative integral or fractional index. The proof of this is difficult and beyond the scope of the present book.

The student is advised to remember the above theorem and to apply it where necessary.

NOTE. The infinite series

$$1+nz+\frac{n(n-1)}{2!}z^2+\frac{n(n-1)(n-2)}{3!}z^3+\ldots$$

is called the Binomial series. It is convergent for all real values of n when $|z|<1$. When $|z|=1$, it is still convergent, if (i) $n>0$, or if (ii) $-1<n<0$ and $z\leqslant -1$. When $n\leqslant -1$, it is divergent.

Ex. By expressing $(1+i)^n$ in two different ways, show that

$$1-{}^nC_2+{}^nC_4-\ldots=2^{n/2}\cos\tfrac{1}{4}n\pi,$$

and

$${}^nC_1-{}^nC_3+{}^nC_5-\ldots=2^{n/2}\sin\tfrac{1}{4}n\pi.$$

Since

$$1+i=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right),$$

therefore, by De Moivre's theorem,

$$(1+i)^n=2^{n/2}\left(\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}\right). \qquad (1)$$

Again, by the Binomial theorem,

$$\begin{aligned}(1+i)^n&=1+{}^nC_1\,i+{}^nC_2\,i^2+{}^nC_3\,i^3\\&\quad+{}^nC_4\,i^4+{}^nC_5\,i^5+\ldots\\&=1+{}^nC_1\,i-{}^nC_2-{}^nC_3\,i\\&\quad+{}^nC_4+{}^nC_5\,i-\ldots\\&=(1-{}^nC_2+{}^nC_4-\ldots)\\&\quad+i({}^nC_1-{}^nC_3+{}^nC_5-\ldots). \qquad (2)\end{aligned}$$

Hence, from (1) and (2), by equating the real and imaginary parts, we have the desired results.

Examples on Chapter III

1. Given that $x=2-2i$, $y=\sqrt{3}-i$, express x and y in the trigonometrical form, and then apply De Moivre's theorem to calculate the expression x^2y^3.

Simplify :

$$\frac{(\cos\theta+i\sin\theta)^8(\cos 2\theta-i\sin 2\theta)^9}{(\cos 2\theta+i\sin 2\theta)^9(\cos\theta-i\sin\theta)^8},$$

where θ is the circular measure of $18°$. [*Birmingham*, 1944]

2. Prove that

$$(a+ib)^{m/n}+(a-ib)^{m/n}$$
$$=2(a^2+b^2)^{m/2n}\cos\{(m/n)\tan^{-1}(b/a)\}.$$ [*Agra*, '53]

3. If $a=\cos 2\alpha+i\sin 2\alpha$, with similar expressions for b, c, d, prove that

(i) $\sqrt{(abc)}+\dfrac{1}{\sqrt{(abc)}}=2\cos(\alpha+\beta+\gamma)$,

(ii) $\sqrt{(ab/cd)}+\sqrt{(cd/ab)}=2\cos(\alpha+\beta-\gamma-\delta)$,

and (iii) $a^pb^qc^rd^s+\dfrac{1}{a^pb^qc^rd^s}=2\cos 2(p\alpha+q\beta+r\gamma+s\delta)$.

4. If $2\cos\theta=a+1/a$ and $2\cos\phi=b+1/b$, prove that $2\cos(\theta+\phi)$ is one of the values of $ab+1/ab$.

[*Allahabad*, 1953]

5. If α, β be the roots of $x^2-2x+4=0$, prove that

$$\alpha^n+\beta^n=2^{n+1}\cos\frac{n\pi}{3}.$$ [*Lucknow*, '68]

6. Find the cube roots of $1-i\sqrt{3}$ and illustrate them on an Argand diagram.

If $p=\cos\theta+i\sin\theta$, $q=\cos\phi+i\sin\phi$, show that

$$(p-q)/(p+q)=i\tan\tfrac{1}{2}(\theta-\phi).$$

7. If n be a positive integer, prove that

$$(\sqrt{3}+i)^n+(\sqrt{3}-i)^n=2^{n+1}\cos\tfrac{1}{6}n\pi.$$ [*Lucknow*, '67]

8. Prove that, if n is a positive integer and

$$(1+x)^n = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n,$$

then $c_0 + c_4 + c_8 + \dots = 2^{n-2} + 2^{n/2-1} \cos \frac{1}{4} n\pi$. [*Lucknow*, 1958]

[Hint. Put $x = 1$, -1 and i successively in the given relation. Thus

$$c_0 + c_1 + c_2 + \dots = 2^n \qquad (1)$$

$$c_0 - c_1 + c_2 - \dots = 0, \qquad (2)$$

$$c_0 + ic_1 - c_2 - ic_3 + \dots = 2^{n/2}(\cos \tfrac{1}{4} n\pi + i \sin \tfrac{1}{4} n\pi). \qquad (3)$$

Add (1) and (2) and divide by 2. Thus

$$c_0 + c_2 + c_4 + \dots = 2^{n-1}. \qquad (4)$$

Equate the real parts on the two sides of (3). Thus

$$c_0 - c_2 + c_4 - \dots = 2^{n/2} \cos \tfrac{1}{4} n\pi. \qquad (5)$$

Finally, add (4) and (5) and divide by 2.]

9. Show that if

$$x = \cos\theta + i \sin\theta \quad \text{and} \quad \sqrt{(1-c^2)} = nc - 1,$$

then $$1 + c\cos\theta = (c/2n)(1+nx)(1+n/x).$$

10. If $\cos(\beta-\gamma) + \cos(\gamma-\alpha) + \cos(\alpha-\beta) = -\frac{3}{2}$, show that

$$\cos\alpha + \cos\beta + \cos\gamma = 0$$

and $$\sin\alpha + \sin\beta + \sin\gamma = 0.$$

11. If $\cos(\beta-\gamma) + \cos(\gamma-\alpha) + \cos(\alpha-\beta) = -\frac{3}{2}$,
prove that $\cos n\alpha + \cos n\beta + \cos n\gamma$ is equal to $3\cos\frac{1}{3}n(\alpha+\beta+\gamma)$ or zero, according as n is or is not a multiple of 3.

[*Banaras*, 1934]

[Hint. Use the following identity :

If $x+y+z=0$, and $1/x+1/y+1/z=0$, then

$x^n+y^n+z^n = 3(xyz)^{n/3}$ or 0, according as n is or is not a multiple of 3.]

12. Use De Moivre's theorem to solve the equations :

(i) $x^4 - x^3 + x^2 - x + 1 = 0$, [*Poona*, 1958]

(ii) $x^7 + x^4 + x^3 + 1 = 0$,

and (iii) $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$.

13. Prove that

$$\sqrt[n]{(a+ib)}+\sqrt[n]{(a-ib)}$$

has n real values and find those of

$$\sqrt[3]{(1+i3)}+\sqrt[3]{(1-i3)}.$$ [*Poona*, 1957; *Luck.*, '61]

14. Find the seven seventh roots of unity and prove that the sum of their nth powers always vanishes unless n be a multiple of 7, n being an integer, and that then the sum is 7. [*Lucknow*, 1966]

15. Use De Moivre's theorem to find all the roots of the equation

$$(2x-1)^5=(x-2)^5$$

in the form $a+ib$, where a and b are real numbers.

16. Prove that every root of the equation

$$(1+x)^6+x^6=0$$

has $-\frac{1}{2}$ for its real part. [*Baroda*, 1953]

17. Solve the equations:

(i) $x^4=1-\sqrt{(-3)}$,

(ii) $4n^4=\sqrt{3}+i$. [*Dacca*, 1950]

18. Find the general value of θ which satisfies the equation

$$(\cos\theta+i\sin\theta)(\cos 3\theta+i\sin 3\theta)\ldots$$
$$\{\cos(2r-1)\theta+i\sin(2r-1)\theta\}=1.$$ [*Cal.*, 1942]

19. From the identity

$$\frac{1}{x-a}-\frac{1}{x-b}=\frac{a-b}{(x-a)(x-b)},$$

deduce that

$$\cos(\theta+\alpha)\sin(\theta-\beta)-\cos(\theta+\beta)\sin(\theta-\alpha)$$
$$=\sin(\alpha-\beta)\cos 2\theta,$$

and
$$\sin(\theta+\alpha)\sin(\theta-\beta)-\sin(\theta+\beta)\sin(\theta-\alpha)$$
$$=\sin(\alpha-\beta)\sin 2\theta.$$

20. Show that the sum of the infinite series

$$1-\frac{1.3}{2.4}+\frac{1.3.5.7}{2.4.6.8}-\frac{1.3.5.7.9.11}{2.4.6.8.10.12}+\ldots$$

is $\frac{1}{2}(1+\sqrt{2})^{1/2}$.

[Hint. Expand $(1+i)^{-1/2}$ by the Binomial Theorem and equate the real part of $(1+i)^{-1/2}$ with that of its expansion.]

CHAPTER IV

IMPORTANT DEDUCTIONS FROM DE MOIVRE'S THEOREM

4·1. Expansions of cos $n\theta$ and sin $n\theta$ (n being a positive integer). When n is a positive integer, we have from De Moivre's theorem

$$\cos n\theta+i\sin n\theta=(\cos\theta+i\sin\theta)^n$$
$$=(\cos\theta)^n+{}^nC_1(\cos\theta)^{n-1}(i\sin\theta)$$
$$+{}^nC_2(\cos\theta)^{n-2}(i\sin\theta)^2+{}^nC_3(\cos\theta)^{n-3}(i\sin\theta)^3+\ldots,\quad(1)$$

on expanding the right hand member by the Binomial Theorem. If we equate the real and imaginary parts on either side, we have

$$\cos n\theta=\cos^n\theta-{}^nC_2\cos^{n-2}\theta\sin^2\theta+{}^nC_4\cos^{n-4}\theta\sin^4\theta-\ldots,$$

and $\sin n\theta={}^nC_1\cos^{n-1}\theta\sin\theta-{}^nC_3\cos^{n-3}\theta\sin^3\theta+{}^nC_5\cos^{n-5}\theta\sin^5\theta-\ldots$

We note that the terms in either series are alternately positive and negative and that the two expansions are true whether n be odd or even, but the last terms of the series are different in the two cases. The last two terms of the expansion (1) are

$${}^nC_{n-1}\cos\theta\,(i\sin\theta)^{n-1}+(i\sin\theta)^n,$$

which will respectively be real and imaginary or imaginary and real according as n is odd or even. For, when n is odd,

$${}^nC_{n-1}\cos\theta(i\sin\theta)^{n-1}=n\cos\theta\,.\,i^{n-1}\sin^{n-1}\theta$$
$$=i^{n-1}n\cos\theta\sin^{n-1}\theta$$
$$=(-1)^{(n-1)/2}n\cos\theta\sin^{n-1}\theta,$$

and $\quad (i\sin\theta)^n = i(i)^{n-1}\sin^n\theta = i(-1)^{(n-1)/2}\sin^n\theta;$

and, when n is even,

$$\begin{aligned} {}^nC_{n-1}\cos\theta\,(i\sin\theta)^{n-1} &= n\cos\theta\,.\,i^{n-1}\sin^{n-1}\theta \\ &= i\,.\,i^{n-2}n\cos\theta\sin^{n-1}\theta, \\ &= i\,(-1)^{(n-2)/2}\,n\cos\theta\sin^{n-1}\theta, \end{aligned}$$

and $\quad (i\sin\theta)^n = i^n\sin^n\theta = (-1)^{n/2}\sin^n\theta.$

Thus the last term in the series for $\cos n\theta$ is

$$(-)^{(n-1)/2}n\cos\theta\sin^{n-1}\theta \text{ or } (-)^{n/2}\sin^n\theta$$

and the last term in the expansion of $\sin n\theta$ is

$$(-)^{(n-1)/2}\sin^n\theta \text{ or } (-)^{(n-2)/2}\,n\cos\theta\sin^{n-1}\theta$$

according as n is odd or even.

4·11. Expansion of tan $n\theta$.

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta}$$

$$= \frac{n\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + ..}{\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + ...}$$

on writing for $\sin n\theta$ and $\cos n\theta$ the series deduced in the previous article. If we divide the numerator and denominator by $\cos^n\theta$, we get

$$\tan n\theta = \frac{n\tan\theta - {}^nC_3\tan^3 + \theta...}{1 - {}^nC_2\tan^2\theta + ...}$$

It is easy to see that, when n is even, the last term of the numerator of $\tan n\theta$ is $n(-1)^{(n-2)/2}$ $\tan^{n-1}\theta$ and the last term of the denominator is $(-1)^{n/2}\tan^n\theta$; and when n is odd, the last term of the numerator is $(-1)^{(n-1)/2}\tan^n\theta$ and the last term of the denominator is $n(-1)^{(n-1)/2}\tan^{n-1}\theta$.

4·2. Expansion of tan $(\theta_1+\theta_2+\theta_3+...+\theta_n)$. We can also easily obtain general formulae for the sine, cosine and tangent of the sum of any number

of angles which are not all equal. We have seen that

$$(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)...(\cos\theta_n+i\sin\theta_n)$$
$$=\cos(\theta_1+\theta_2+...+\theta_n)+i\sin(\theta_1+\theta_2+...+\theta_n).$$

Now

$$(\cos\theta_1+i\sin\theta_1)=\cos\theta_1(1+i\tan\theta_1),$$
$$(\cos\theta_2+i\sin\theta_2)=\cos\theta_2(1+i\tan\theta_2),$$

and so on. Hence

$$(\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2)...(\cos\theta_n+i\sin\theta_n)$$
$$=\cos\theta_1\cos\theta_2...\cos\theta_n(1+i\tan\theta_1)(1+i\tan\theta_2)+...$$
$$+(1+i\tan\theta_n)$$
$$=\cos\theta_1\cos\theta_2...\cos\theta_n[1+is_1-s_2-is_3+s_4+...],$$

where $s_1=\tan\theta_1+\tan\theta_2+...+\tan\theta_n$,

$$s_2=\tan\theta_1\tan\theta_2+\tan\theta_2\tan\theta_3+...,$$

and so on.

It follows that

$$\cos(\theta_1+\theta_2+...+\theta_n)+i\sin(\theta_1+\theta_2+...+\theta_n)$$
$$=\cos\theta_1\cos\theta_2...\cos\theta_n[1+is_1-s_2-is_3+s_4+...].$$

Equating real and imaginary parts on either side, we have

$$\cos(\theta_1+\theta_2+...+\theta_n)=\cos\theta_1\cos\theta_2...\cos\theta_n\times(1-s_2+s_4-...), \quad (1)$$

and $$\sin(\theta_1+\theta_2+...+\theta_n)=\cos\theta_1\cos\theta_2...\cos\theta_n\times(s_1-s_3+s_5-...). \quad (2)$$

Dividing either side of (2) by the corresponding side of (1), we get

$$\tan(\theta_1+\theta_2+...+\theta_n)=\frac{s_1-s_3+s_5-...}{1-s_2+s_4-...}. \quad (3)$$

We can easily see that the last terms of the numerator and denominator will respectively be

$(-1)^{(n-1)/2}\, s_{n-1}$ and $(-1)^{n/2}\, s_n$, when n is even, and $(-1)^{(n-1)/2}\, s_n$ and $(-1)^{(n-1)/2}\, s_{n-1}$, when n is odd.

NOTE. If in (1), (2) and (3), we put

$$\theta_1=\theta_2=\ldots=\theta_n=\theta$$

we can easily deduce the formulae for $\sin n\theta$, $\cos n\theta$ and $\tan n\theta$, given above.

Ex. 1. Show that

$$\tan\frac{\theta}{n}+\tan\frac{\theta+\pi}{n}+\tan\frac{\theta+2\pi}{n}+\ldots+\tan\frac{\theta+(n-1)\pi}{n}$$
$$=n\tan\theta \text{ or } -n\cot\theta,$$

according as n is odd or even.

Let $\quad a=\dfrac{\theta+r\pi}{n}$, where $r=0, 1, 2, 3, \ldots$.

[This assumption is based on the fact that the angles of the given series of tangents are in an arithmetical progression whose general term is $(\theta+r\pi)/n$.]

Then $\quad na=\theta+r\pi.$

Therefore $\quad \tan na=\tan\theta. \quad (1)$

When n is odd, this equation can be written as

$$\frac{nt-{}^nC_3t^3+\ldots+(-)^{(n-1)/2}\,t^n}{1-{}^nC_2t^2+\ldots+n(-)^{(n-1)/2}\,t^{n-1}}=\tan\theta,$$

where $t\equiv\tan a$; or,

$$t^n-nt^{n-1}\tan\theta+\ldots=0.$$

The roots of this equation are

$\tan\dfrac{\theta}{n}, \tan\dfrac{\theta+\pi}{n}, \ldots, \tan\dfrac{\theta+(n-1)\pi}{n}$; for, $t\equiv\tan a$ and

$$a=\frac{\theta}{n}, \frac{\theta+\pi}{n}, \frac{\theta+2\pi}{n}, \ldots.$$

Therefore, by the Theory of Equations, the sum of the roots,

i.e., $\quad \tan\dfrac{\theta}{n}+\tan\dfrac{\theta+\pi}{n}+\ldots+\tan\dfrac{\theta+(n-1)\pi}{n}=n\tan\theta.$

When n is even, (1) can be written as

$$\frac{nt-{}^nC_3t^3+\ldots+n(-1)^{(n-2)/2}\,t^{n-1}}{1-{}^nC_2\,t^2+\ldots+(-)^{n/2}\,t^n}=\tan\theta,$$

or

$$t^n\tan\theta+nt^{n-1}+\ldots=0.$$

Therefore, in this case, the sum of the roots, i.e.,

$$\tan\frac{\theta}{n}+\tan\frac{\theta+\pi}{n}+\ldots+\tan\frac{\theta+(n-1)\pi}{n}=-n\cot\theta.$$

Ex. 2. Show that $\cos\frac{r\pi}{13}$, $r=1, 3, 5, 7, 9, 11$, are the roots of the equation

$$64x^6-32x^5-80x^4+32x^3+24x^2-6x-1=0.$$

[*Dacca*, 1951]

Let $\theta=\frac{r\pi}{13}$, $r=1, 3, 5, 7, 11, \ldots$,

so that $\theta=\frac{(2n+1)\pi}{13}$, $n=0, 1, 2, \ldots$,

Then $13\theta=(2n+1)\pi$,

or $7\theta=(2n+1)\pi-6\theta$,

therefore, $\cos 7\theta=\cos\{(2n+1)\pi-6\theta\}=-\cos 6\theta.$ (1)

Now

$$\begin{aligned}\cos 7\theta+\cos\theta&=2\cos 4\theta\cos 3\theta\\&=2(2\cos^2 2\theta-1)(4\cos^3\theta-3\cos\theta)\\&=2\{2(2\cos^2\theta-1)^2-1\}(4\cos^3\theta-3\cos\theta)\\&=2\{2(2x^2-1)^2-1\}(4x^3-3x),\text{ writing }\cos\theta=x,\\&=2(8x^4-8x^2+1)(4x^3-3x)\\&=64x^7-112x^5+56x^3-6x.\end{aligned}$$

Therefore $\cos 7\theta=64x^7-112x^5+56x^3-7x.$

Also
$$\begin{aligned}\cos 6\theta&=2\cos^2 3\theta-1\\&=2(4\cos^3\theta-3\cos\theta)^2-1\\&=2(4x^3-3x)^2-1\\&=32x^6-48x^4+18x^2-1.\end{aligned}$$

Hence from (1), we get that

$$64x^7-112x^5+56x^3-7x=-(32x^6-48x^4+18x^2-1)$$

or $64x^7+32x^6-112x^5-48x^4+56x^3+18x^2-7x-1=0$

or $(x+1)(64x^6-32x^5-80x^4+32x^3+24x^2-6x-1)=0$

has for its roots $\cos\frac{r\pi}{13}$, where $r=1, 3, 5, 7, 9, 11, 13$.

Omitting the factor $x+1$ corresponding to $r=13$, we get that $\cos\frac{r\pi}{13}$, $r=1, 3, 5, 7, 9, 11$, are the roots of

$$64x^6-32x^5-80x^4+32x^3+24x^2-6x-1=0.$$

Ex. 3. Find the equation whose roots are

$$s_1^2,\ s_3^2,\ s_5^2,\ s_7^2,\ s_9^2,\ s_{11}^2,$$

where $s_r = \sin\dfrac{r\pi}{26}$.

From Ex. 2, $x \equiv \cos\dfrac{r\pi}{13}$, $r=1, 3, 5, 7, 9, 11$, are the roots of

$$64x^6-32x^5-80x^4+32x^3+24x^2-6x-1=0. \qquad (1)$$

We are required to find the equation where roots are $y \equiv \sin^2\dfrac{r\pi}{26}$, where $r=1, 3, 5, 7, 9, 11$.

Now $$\sin^2\frac{r\pi}{26}=\tfrac{1}{2}\left(1-\cos\frac{r\pi}{13}\right)$$

$\therefore$ $$\cos\frac{r\pi}{13}=1-2\sin^2\frac{r\pi}{26}$$

i.e., $$x=1-2y.$$

Therefore, substituting $1-2y$ for x in (1), the required equation is

$$64(1-2y)^6-32(1-2y)^5-80(1-2y)^4+32(1-2y)^3 +24(1-2y)^2-6(1-2y)-1=0$$

i.e., $4096y^6-11264y^5+11520y^4-5376y^3+1120y^2-84y+1=0.$

Ex. 4. Prove that

(i) $$\tan^2\frac{\pi}{11}+\tan^2\frac{2\pi}{11}+\tan^2\frac{3\pi}{11}+\tan^2\frac{4\pi}{11}+\tan^2\frac{5\pi}{11}=55,$$

(ii) $$\tan\frac{\pi}{11}\tan\frac{2\pi}{11}\tan\frac{3\pi}{11}\tan\frac{4\pi}{11}\tan\frac{5\pi}{11}=\sqrt{11}.$$

Let $\theta=\dfrac{n\pi}{11}$, where $n=1, 2, 3, \ldots$. Then

$$11\theta=n\pi,$$

so that $$\tan 11\theta=0.$$

Therefore expanding $\tan 11\theta$ in powers of $\tan\theta$, we get that $\tan\dfrac{r\pi}{11}$, $r=1, 2, 3, \ldots, 11$, are the roots of the equation

$$11 \tan\theta - {}^{11}C_3 \tan^3\theta + {}^{11}C_5 \tan^5\theta - {}^{11}C_7 \tan^7\theta + {}^{11}C_9 \tan^9\theta - \tan^{11}\theta = 0$$

i.e., $$\tan^{11}\theta - 55\tan^9\theta + 330\tan^7\theta - 462\tan^5\theta + 165\tan^3\theta - 11\tan\theta = 0$$

i.e., $$\tan\theta(\tan^{10}\theta - 55\tan^8\theta + 330\tan^6\theta - 462\tan^4\theta + 165\tan^2\theta - 11) = 0.$$

The factor $\tan\theta$ corresponds to $r=11$. Therefore the roots of the equation
$$\tan^{10}\theta - 55\tan^8\theta + 330\tan^6\theta - 462\tan^4\theta + 165\tan^2\theta - 11 = 0$$
are $\tan\dfrac{r\pi}{11}$, $r=1, 2, 3, \ldots, 10$.

Now putting $\tan^2\theta = x$, we obtain the equation
$$x^5 - 55x^4 + 330x^3 - 462x^2 + 165x - 11 = 0,$$
the roots of which are $\tan^2\dfrac{r\pi}{11}$, $r=1, 2, 3, 4, 5$, because

$$\tan^2\frac{6\pi}{11} = \tan^2\frac{5\pi}{11},\ \tan^2\frac{7\pi}{11} = \tan^2\frac{4\pi}{11},\ \tan^2\frac{8\pi}{11} = \tan^2\frac{3\pi}{11},$$
$$\tan^2\frac{9\pi}{11} = \tan^2\frac{2\pi}{11},\ \text{and}\ \tan^2\frac{10\pi}{11} = \tan^2\frac{\pi}{11}.$$

Therefore, by the Theory of Equations, we have

$$\tan^2\frac{\pi}{11} + \tan^2\frac{2\pi}{11} + \tan^2\frac{3\pi}{11} + \tan^2\frac{4\pi}{11} + \tan^2\frac{5\pi}{11} = 55; \quad (1)$$

and $$\tan^2\frac{\pi}{11}\tan^2\frac{2\pi}{11}\tan^2\frac{3\pi}{11}\tan^2\frac{4\pi}{11}\tan^2\frac{5\pi}{11} = 11,$$

i.e., $$\tan\frac{\pi}{11}\tan\frac{2\pi}{11}\tan\frac{3\pi}{11}\tan\frac{4\pi}{11}\tan\frac{5\pi}{11} = \sqrt{11}. \quad (2)$$

Ex. 5. Prove that the equation
$$\sin 3\theta = a\sin\theta + b\cos\theta + c$$
has six roots and that the sum of the six values of θ, which satisfy it, is equal to an odd multiple of π radians.

Let $t \equiv \tan(\theta/2)$. Then since
$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta,$$
$$\sin\theta = 2\tan(\theta/2)/\{1+\tan^2(\theta/2)\}$$
and $$\cos\theta = \{1-\tan^2(\theta/2)\}/\{1+\tan^2(\theta/2)\},$$
therefore the above equation becomes

$$\frac{6t}{1+t^2}-4\left\{\frac{2t}{1+t^2}\right\}^3=\frac{2at}{1+t^2}+\frac{b(1-t^2)}{1+t^2}+c,$$

which simplifies to

$$(b-c)t^6-2(a-3)t^5+(b-3c)t^4-4(a+5)t^3 \\ -(b+3c)t^2-2(a-3)t-(b+c)=0.$$

This is a sixth degree equation in t and so it has 6 roots. Let the six roots be $t=\tan(\theta/2)$, $\theta=\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6$. Then by the Theory of Equations,

$$s_1=2(a-3)/(b-c),$$
$$s_2=(b-3c)/(b-c),$$
$$s_3=4(a+5)/(b-c),$$
$$s_4=-(b+3c)/(b-c)$$
$$s_5=2(a-3)/(b-c).$$

and $$s_6=-(b+c)/(b-c),$$

where s_1, s_2, s_3, etc., have their usual meanings.

Therefore

$$\tan \tfrac{1}{2}(\theta_1+\theta_2+\ldots+\theta_6)=\frac{s_1-s_3+s_5}{1-s_2+s_4-s_6}=\infty=\tan(n\pi+\pi/2)$$
$$=\tan(2n+1)\pi/2.$$

Hence $$\theta_1+\theta_2+\ldots+\theta_6=(2n+1)\pi.$$

Examples

Prove that

1. $\sin 4\theta=4(\cos^3\theta\sin\theta-\cos\theta\sin^3\theta)$.
2. $\cos 5\theta=\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta$.
3. $\cos 6\theta=\cos^6\theta-15\cos^4\theta\sin^2\theta+15\cos^2\theta\sin^4\theta-\sin^6\theta$.
4. $\sin 7\theta=7\cos^6\theta\sin\theta-35\cos^4\theta\sin^3\theta$
$+21\cos^2\theta\sin^5\theta-\sin^7\theta$.
5. $\tan 4\theta=\dfrac{4(\tan\theta-\tan^3\theta)}{1-6\tan^2\theta+\tan^4\theta}$.
6. $\tan 7\theta=\dfrac{7\tan\theta-35\tan^3\theta+21\tan^5\theta-\tan^7\theta}{1-21\tan^2\theta+35\tan^4\theta-7\tan^6\theta}$.
7. If α, β and γ be the roots of the equation
$$x^3+px^2+qx+p=0,$$

prove that $\tan^{-1}\alpha+\tan^{-1}\beta+\tan^{-1}\gamma=n\pi$ radians, except in one particular case. [*Lucknow*, 1962; *Alld.*, 1953]

8. Write down the five values of θ that lie in the interval $\left(-\frac{\pi}{2}\leqslant\theta<\frac{\pi}{2}\right)$ and conform to the equation

$$\sin 5\theta=0.$$

Deduce or prove directly that the four roots of the equation

$$16x^4-20x^2+5=0$$

are $\pm\sin\frac{\pi}{5}$ and $\pm\sin\frac{2\pi}{5}$. [*Gauhati*, 1950]

9. Prove that the roots of the equation

$$8x^3+4x^2-4x-1=0$$

are $\cos\frac{2\pi}{7}, \cos\frac{4\pi}{7}, \cos\frac{6\pi}{7}$. [*Lucknow*, '63]

10. Show that $\cos\frac{\pi}{7}$, $\cos\frac{3\pi}{7}$, and $\cos\frac{5\pi}{7}$ are the roots of the equation $8x^3-4x^2-4x+1=0$, and deduce the equation whose roots are

$$\tan^2\frac{\pi}{7},\quad \tan^2\frac{3\pi}{7}\quad\text{and}\quad \tan^2\frac{5\pi}{7}.$$ [*Baroda*, 1952]

11. Show that $\cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, \cos\frac{6\pi}{9}, \cos\frac{8\pi}{9}$ are the roots of

$$16x^4+8x^3-12x^2-4x+1=0.$$ [*Travancore*, 1940]

12. Form the equation whose roots are

$$\cos\frac{\pi}{11}, \cos\frac{3\pi}{11}, \cos\frac{5\pi}{11}, \cos\frac{7\pi}{11}, \cos\frac{9\pi}{11},$$

and hence find the values of

(i) $\cos\frac{\pi}{11}+\cos\frac{3\pi}{11}+\cos\frac{5\pi}{11}+\cos\frac{7\pi}{11}+\cos\frac{9\pi}{11}$,

(ii) $\sec\frac{\pi}{11}+\sec\frac{3\pi}{11}+\sec\frac{5\pi}{11}+\sec\frac{7\pi}{11}+\sec\frac{9\pi}{11}$.

[*Andhra*, 1940]

13. Find the equation whose roots are

$$\pm\tan\tfrac{1}{5}\pi\quad\text{and}\quad\pm\tan\tfrac{2}{5}\pi.$$

14. By means of the equation $\tan 7\theta = 0$, or otherwise, find the value of

$$\tan\frac{\pi}{7}\tan\frac{2\pi}{7}\tan\frac{3\pi}{7} \text{ and } \cot^2\frac{\pi}{7}+\cot^2\frac{2\pi}{7}+\cot^2\frac{3\pi}{7}.$$

15. Prove that

$$\operatorname{cosec}^2\frac{\pi}{7}\operatorname{cosec}^2\frac{2\pi}{7}+\operatorname{cosec}^2\frac{2\pi}{7}\operatorname{cosec}^2\frac{3\pi}{7}+\operatorname{cosec}^2\frac{3\pi}{7}\operatorname{cosec}^2\frac{\pi}{7}=16.$$

16. Form an equation whose roots are

$$\tan^2\frac{\pi}{9}, \tan^2\frac{2\pi}{9}, \tan^2\frac{3\pi}{9}, \tan^2\frac{4\pi}{9}.$$

17. Prove that

$$\cot^2\frac{\pi}{11}+\cot^2\frac{2\pi}{11}+\cot^2\frac{3\pi}{11}+\cot^2\frac{4\pi}{11}+\cot^2\frac{5\pi}{11}=15.$$

[*Andhra*, 1935]

18. Prove that if n be an integer

$$\cot\alpha+\cot\left(\alpha+\frac{\pi}{n}\right)+\cot\left(\alpha+\frac{2\pi}{n}\right)+\ldots\text{to } n \text{ terms}$$
$$= n\cot n\alpha.$$

[*Cal.*, 1937]

19. If α, β, γ and δ are the roots of the equation $\tan(\pi/4+\theta) = 3\tan 3\theta$, no two of which have equal tangents, show that

$$\tan\alpha+\tan\beta+\tan\gamma+\tan\delta=0.$$

[*Andhra*, 1937]

20. Prove that the equation

$$a^2\cos^2\theta+b^2\sin^2\theta+2ag\cos\theta+2bf\sin\theta+c=0$$

has four roots and that the sum of the values of θ which satisfy it is an even multiple of π radians.

21. Solve the equation

$$\cot 3\theta = 1,$$

subject to the condition $0<\theta<\pi$. Hence using the formula for $\cot 3\theta$ in terms of $\cot\theta$, prove that the three roots of the cubic $x^3-3x^2-3x+1=0$ are

$$\cot\frac{\pi}{12}, \cot\frac{5\pi}{12}, \text{ and } \cot\frac{3\pi}{4}.$$

[*Cal.*, 1942]

22. If S_n denote the sum of the products of $\tan^2\theta_1$, $\tan^2\theta_2$, ... taken n together and s_n denote the sum of the products of $\tan\theta_1$, $\tan\theta_2$, ... taken n together, show that

$$1+S_1+S_2+\ldots=(1-s_2+s_4-\ldots)^2+(s_1-s_3+\ldots)^2.$$

[HINT. Square and add (1) and (2) of § 4·2.]

23. If $s_1, s_2, \ldots, s_n$ be the sum of the products of the n quantities $\tan\alpha$, $\tan 2\alpha$, $\tan 2^2\alpha$, ..., $\tan 2^{n-1}\alpha$, taken 1, 2, 3, ..., n together, show that

$$1-s_2+s_4-\ldots=2^n\sin\alpha\cos(2^n-1)\alpha\operatorname{cosec}2^n\alpha,$$

and
$$s_1-s_3+\ldots=2^n\sin\alpha\sin(2^n-1)\alpha\operatorname{cosec}2^n\alpha.$$

4·3. Series for sin α. We know from § 4·1 that, when n is a positive integer,

$$\sin n\theta=n\cos^{n-1}\theta\sin\theta-\frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta\sin^3\theta+\ldots.$$

Let $n\theta=\alpha$ and suppose that n increases without limit and θ decreases in such a way that $n\theta=\alpha$. The above equation may be written as

$$\sin\alpha=\alpha\cos^{n-1}\theta\,\frac{\sin\theta}{\theta}-\frac{\alpha(\alpha-\theta)(\alpha-2\theta)}{3!}\cos^{n-3}\theta\left(\frac{\sin\theta}{\theta}\right)^3$$
$$+\frac{\alpha(\alpha-\theta)(\alpha-2\theta)(\alpha-3\theta)(\alpha-4\theta)}{5!}\cos^{n-5}\theta\left(\frac{\sin\theta}{\theta}\right)^5-\ldots.$$

When θ decreases without limit, $\dfrac{\sin\theta}{\theta}$ and $\cos\theta$ each tend to unity and so does every power of these quantities. Hence the above formula becomes

$$\sin\alpha=\alpha-\frac{\alpha^3}{3!}+\frac{\alpha^5}{5!}-\ldots.$$

This is, of course, an infinite series and its convergence can easily be tested by the help of D' Alembert's ratio-test. Thus, since the general term of the above series is

$$u_n = (-)^{n-1} \frac{\alpha^{2n-1}}{(2n-1)!},$$

therefore, $$\frac{u_{n+1}}{u_n} = -\frac{\alpha^2}{2n(2n+1)},$$

giving that

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = 0$$

for all values of α. It follows that the series for $\sin \alpha$ is absolutely convergent for all values of α.

It may, however, be noted that the above proof depends upon the limit $\frac{\sin \theta}{\theta}$ being unity when θ tends to zero and this is true only when θ is given in radians. Hence the series-expansion for $\sin \alpha$ is true when α is given in radians. The formula may be modified when other unit of angular measurement is adopted. Thus for the expansion of $\sin n°$, we have to convert n degrees into radians and if α be its equivalent,

$$\alpha = \frac{n\pi}{180}.$$

Hence

$$\sin n° = \sin\left(\frac{n\pi}{180}\right)$$

$$= \frac{n\pi}{180} - \frac{1}{3!}\left(\frac{n\pi}{180}\right)^3 + \frac{1}{5!}\left(\frac{n\pi}{180}\right)^5 - \dots.$$

4·31. Series for cos α. Again if we use the formula

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{2!}\cos^{n-2}\theta\sin^2\theta$$
$$+\frac{n(n-1)(n-2)(n-3)}{4!}\cos^{n-4}\theta\sin^4\theta - \ldots,$$

and proceed as above, we shall get

$$\cos\alpha = 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \ldots.$$

The general term of this series is given by

$$u_n = (-)^{n-1}\frac{\alpha^{2n-2}}{(2n-2)!},$$

and its absolute convergence follows by D'Alembert's ratio-test. Moreover, we have

$$\cos n^\circ = \cos\frac{n\pi}{180}$$
$$= 1 - \frac{1}{2!}\left(\frac{n\pi}{180}\right)^2 + \frac{1}{4!}\left(\frac{n\pi}{180}\right)^4 - \ldots.$$

The above expansions for sin α and cos α are of very great importance.

Ex. 1. Given $\frac{\sin\theta}{\theta} = \frac{2165}{2166}$, show that θ is nearly the circular measure of 3°.

We know that $\frac{\sin\theta}{\theta}$ tends to unity when θ tends to zero.

Here $\frac{\sin\theta}{\theta}$ is nearly equal to unity, hence θ must be small.

But

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$$

giving
$$\frac{\sin\theta}{\theta}=1-\frac{\theta^2}{3!}+\frac{\theta^4}{5!}-\dots.$$

Neglecting θ^4 and higher powers of θ, we have
$$\frac{\sin\theta}{\theta}=1-\frac{\theta^2}{6}=\frac{2165}{2166},$$
or
$$\theta^2=6\left(1-\frac{2165}{2166}\right)=\frac{1}{361}.$$

Therefore, $\theta=\frac{1}{19}$ radians $=\frac{180}{\pi}\cdot\frac{1}{19}$ degrees
$$=3^\circ \text{ nearly.}$$

Ex. 2. If $\cos(\frac{1}{3}\pi+x)=\cdot 49$, show that
$$x=39'\cdot 5 \text{ approx.}$$

Since $\cos(\frac{1}{3}\pi+x)$ is equal to $\cdot 49$, i.e., very nearly equal to $\cdot 5$, therefore x is very small, so that we may write $\sin x=x$ and $\cos x=1$ nearly.

Now
$$\cos(\tfrac{1}{3}\pi+x)=\cdot 49$$
$\therefore$ $$\cos(\tfrac{1}{3}\pi)\cos x-\sin(\tfrac{1}{3}\pi)\sin x=\cdot 49$$
i.e., $$\tfrac{1}{2}-(\sqrt{3}/2)x=\cdot 49$$
i.e., $$\cdot 5-\cdot 87\,x=\cdot 49$$
$\therefore$ $$x=\frac{\cdot 01}{\cdot 87}\text{ radians}$$
$$=\frac{3437\cdot 7}{87}\text{ minutes, [1 radian} =3437'\cdot 7]$$
$$=39'\cdot 5.$$

Ex. 3. Assuming the expansions of $\sin x$ and $\cos x$ in powers of x, adjust the constants a and b in such a way that
$$\lim_{x\to 0}\frac{a\cos x+bx\sin x-5}{x^4}$$
may exist. Also find this limit when a, b are so adjusted.

[*Cal.*, 1942]

We have
$$\frac{a\cos x+bx\sin x-5}{x^4}$$

$$=\frac{a[1-x^2/2!+x^4/4!-x^4/6!+\ldots]+bx[x-x^3/3!+x^5/5!-\ldots]-5}{x^4}$$

$$=\frac{a-5}{x^4}+\frac{b-\frac{1}{2}a}{x^2}+\frac{a-4b}{24}+\text{terms involving } x^2 \text{ and higher powers of } x.$$

If this has a limit, as x tends to zero, we must have $a-5=0$ and $b-\frac{1}{2}a=0$, i.e., $a=5$ and $b=5/2$; and in that case the required limit is $(a-4b)/24$, i.e., $-5/24$.

Ex. 4. Prove that

$$\sin^2\theta\cos\theta=\theta^2-\tfrac{5}{6}\theta^4+\ldots+(-1)^{n+1}\frac{3^{2n}-1}{4}\cdot\frac{\theta^{2n}}{(2n)!}+\ldots.$$

[*Cal.*, 1948]

We have

$$\sin^2\theta\cos\theta=\sin\theta\,.\cos\theta\,.\sin\theta$$

$$=\tfrac{1}{2}\sin 2\theta\,.\sin\theta$$

$$=\tfrac{1}{4}(\cos\theta-\cos 3\theta)$$

$$=\tfrac{1}{4}\left[\left\{1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots+(-1)^n\frac{\theta^{2n}}{(2n)!}+\ldots\right\}\right.$$

$$\left.-\left\{1-\frac{(3\theta)^2}{2!}+\frac{(3\theta)^4}{4!}-\ldots+(-1)^n\frac{(3\theta)^{2n}}{(2n)!}+\ldots\right\}\right]$$

$$=\tfrac{1}{4}\left[4\theta^2-\frac{20\theta^4}{6}+\ldots+(-1)^{n+1}\frac{3^{2n}-1}{(2n)!}\cdot\theta^{2n}+\ldots\right]$$

$$=\theta^2-\tfrac{5}{6}\theta^4+\ldots+(-1)^{n+1}\frac{3^{2n}-1}{4}\cdot\frac{\theta^{2n}}{(2n)!}+\ldots.$$

Examples

1. Find θ approximately to the nearest minute, if $\sin\theta/\theta=5045/5046$. [*Andhra*, 1943]

2. If $\sin\theta=(5765/5766)\theta$, show that $\theta=1°51'$ approx. [*Dacca*, 1950]

3. If $\cos\theta=\theta$, show that $\theta=\cdot 7$ nearly.

4. Find θ approximately to the nearest minute, if $\cos\theta=1681/1682$, taking $\pi=3{\cdot}14159$. [*Andhra*, 1938]

5. If $\sin\theta=\cdot 5033$, show that θ is approximately equal to $30°13'6''$. (1 radian $=206265''$) [*Dacca*, 1951]

6. If $(\sin x)/x = 19493/19494$, show that x is equal to $1°$ approximately.

7. $\cos\theta = \cdot 72136$, $\sqrt{2} = 1\cdot 414...$, show that θ is approximately equal to $43°51'$.

8. Given that $\sin(\pi/4 + 2x) = \cdot 706$, find an approximate value of x.

9. If $\cos(\pi/3 - x) = \cdot 5081$, prove that $x = 32'$ approx.

10. Evaluate the following :

(i) $\lim_{x\to 0} \dfrac{e^x - 1 + \log(1-x)}{x^3}$. [*Madras*, 1936]

(ii) $\lim_{x\to 0} \dfrac{3\sin x - \sin 3x}{x - \sin x}$. [*Poona*, 1956]

(iii) $\lim_{\theta\to 0} \dfrac{\sin n\theta - n\sin\theta}{\theta(\cos n\theta - \cos\theta)}$. [*Andhra*, 1935]

(iv) $\lim_{x\to 0} \dfrac{\sin x\log(1+x) - 4\sin^2(x/2)}{x^3}$. [*Travan.*, 1940]

(v) $\lim_{x\to 0} \dfrac{1 - \cos x°}{x\sin x}$. [*Dacca*, 1935]

(vi) $\lim_{x\to 0} \dfrac{\tan 2x - 2\sin x}{x^3}$. [*Annam.*, 1947]

(vii) $\lim_{x\to 0} \dfrac{\tan 2x - 2\tan x}{2x}$. [*Cal.*, 1933]

(viii) $\lim_{x\to 0} \{(\tan x)/x\}^{1/x}$.

(ix) $\lim_{x\to 0} \{(\tan x)/x\}^{3/x^2}$.

(x) $\lim_{x\to 0} (\cos x)^{1/x}$. [*Bombay*, 1947]

(xi) $\lim_{x\to 0} \dfrac{1 + \sin x - \cos x}{\sin x + \cos x - 1}$.

(xii) $\lim_{\theta\to \pi/2} \dfrac{1 - \sin\theta}{\cos^2\theta - 2\cos^2 3\theta}$. [*Madras*, 1940]

(xiii) $\lim_{x\to \pi/2} (\sec x - \tan x)$. [*Annam.*, 1952]

(xiv) $\lim_{\alpha\to \pi/2} \dfrac{1 - \sin\alpha}{3\cos^2\alpha - \cos^2 3\alpha}$. [*Annam.*, 1943]

(xv) $\lim_{x\to \pi/2} \dfrac{\cot^2 x - \sin^2 2x}{(x - \pi/2)^2}$. [*Andhra*, 1935]

(xvi) $\lim_{x\to a} \cot(x-a)\log(2-x/a)$. [*Bom.*, 1947]

(xvii) $\lim_{x\to\pi/4}(1-\tan x)/(1-\sqrt{2}\sin x)$. [*Bom.*, 1936]

(xviii) $\lim_{n\to\infty}\{\cos(x/n)\}^n$.

(xix) $\lim_{n\to\infty}\{\cos(x/n)\}^{n^2}$.

(xx) $\lim_{n\to\infty}\{\cos(x/n)\}^{n^3}$.

(xxi) $\lim_{x\to\pi/2}(\sin x)^{\tan x}$.

(xxii) $\lim_{x\to 0}(\cos x)^{\cot^2 x}$.

11. Determine the quantities a, b, c so that as θ tends to zero, the expression

$$\frac{\theta(a+b\cos\theta)-c\sin\theta}{\theta^5}$$

may tend to unity. [*Annam.*, 1950]

12. Sum to infinity

$$2x-\frac{4}{3!}x^3+\frac{6}{5!}x^5-\frac{8}{7!}x^7+\dots .$$

13. Expand the following in powers of θ:

$$\sin^2\theta,\ \cos^3\theta, \text{ and } \tan\theta.$$

14. Expand the following in powers of θ and find the general term:

(i) $\{(\sin\theta)/\theta\}^3$. [*Luck.*, '59] (ii) $\sin\theta\cos^2\theta$.

15. Prove that

$$\tfrac{1}{6}\sin^3\theta=\frac{\theta^3}{3!}-(1+3^2)\frac{\theta^5}{5!}+(1+3^2+3^4)\frac{\theta^7}{7!}-\dots . \quad [\textit{Agra}, 1938]$$

16. If θ be small, prove that

$$\theta\cot\theta=1-\tfrac{1}{3}\theta^2-\tfrac{1}{45}\theta^4 \text{ approx.}$$

17. Show that, if θ is very small, the expression

$$\frac{3\sin 2\theta}{2(2+\cos 2\theta)}$$

differs from θ by $(4/45)\theta^5$ nearly. [*Travancore*, 1953]

18. Show that

$$\frac{\pi^2}{2.4}-\frac{\pi^4}{2.4.6.8}+\frac{\pi^6}{2.4.6.8.10.12}-\dots=1.$$

19. If $\tan x = a_1 x + \frac{a_3}{3!} x^3 + \frac{a_5}{5!} x^5 + \ldots$, show that

$$a_{2n+1} = \frac{(2n+1)2n}{2!} a_{2n-1} - \frac{(2n+1)2n(2n-1)(2n-2)}{4!} a_{2n-3} + \ldots + (-1)^{n+1}(2n+1)a_1 + (-1)^n.$$

[*Cal.*, 1946]

20. If $\sec \theta = a_0 + a_2 \theta^2 + a_4 \theta^4 + \ldots + a_{2n}\theta^{2n} + \ldots$, show that

$$a_{2n} = \frac{a_{2n-2}}{2!} - \frac{a_{2n-4}}{4!} + \ldots + \frac{(-1)^{n+1}}{(2n)!} a_0.$$

CHAPTER V

TRIGONOMETRICAL AND EXPONENTIAL FUNCTIONS OF COMPLEX QUANTITIES

5·1. The exponential functions. The student is already familiar with the exponential function

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \ldots, \qquad (1)$$

when x is real.

We now extend this for the complex quantity $z \equiv x + iy$ and define the exponential function of z as

$$E(z) = 1 + z + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots . \qquad (2)$$

We can easily investigate the convergence of this infinite series of powers of z. By applying D'Alembert's ratio test, we see that the series

$$1 + r + \frac{r^2}{2!} + \ldots + \frac{r^n}{n!} + \ldots,$$

where r is real and positive, is convergent for all values of r. If now we form another series from it by changing the signs of some of the terms or by diminishing the values of some of the terms, then this new series will also be convergent. It follows that the series

$$1 + r\cos\theta + \frac{r^2}{2!}\cos 2\theta + \ldots + \frac{r^n}{n!}\cos n\theta + \ldots$$

and $$r \sin \theta + \frac{r^2}{2!} \sin 2\theta + \ldots + \frac{r^n}{n!} \sin n\theta + \ldots$$

are convergent. In fact, these are absolutely convergent for all values of r and θ, since for all n

$$|\cos n\theta| \leqslant 1 \text{ and } |\sin n\theta| \leqslant 1.$$

Now suppose that

$$z = x + iy = r(\cos \theta + i \sin \theta).$$

Then

$$E(z) = 1 + z + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots$$

$$= 1 + \frac{r(\cos \theta + i \sin \theta)}{1!} + \frac{r^2(\cos \theta + i \sin \theta)^2}{2!} + \ldots$$

$$= 1 + \frac{r}{1!}(\cos \theta + i \sin \theta) + \frac{r^2}{2!}(\cos 2\theta + i \sin 2\theta) + \ldots$$

on using De Moivre's theorem,

$$= \{1 + \frac{r \cos \theta}{1!} + \frac{r^2}{2!} \cos 2\theta + \ldots\}$$

$$+ i\{r \sin \theta + \frac{r^2}{2!} \sin 2\theta + \ldots\},$$

on re-arranging the terms.

But the two series within brackets have just been shown to be absolutely convergent. Hence the series defining the exponential function $E(z)$ is absolutely convergent for all values of z.

5·11. Product of exponential functions. *To prove that* $E(z_1) \times E(z_2) = E(z_1 + z_2)$.

From the above definition,

$$E(z_1)\times E(z_2)=\left(1+z_1+\frac{z_1^{\,2}}{2!}+\ldots+\frac{z_1^{\,n}}{n!}+\ldots\right)$$
$$\times\left(1+z_2+\frac{z_2^{\,2}}{2!}+\ldots+\frac{z_2^{\,n}}{n!}+\ldots\right)$$
$$=1+(z_1+z_2)+\left(\frac{z_1^{\,2}}{2!}+z_1z_2+\frac{z_2^{\,2}}{2!}\right)+\ldots$$
$$+\left(\frac{z_1^{\,n}}{n!}+\frac{z_1^{\,n-1}}{(n-1)!}\cdot\frac{z_2}{1!}+\frac{z_1^{\,n-2}}{(n-2)!}\cdot\frac{z_2^{\,2}}{2!}+\ldots+\frac{z_2^{\,n}}{n!}\right)+\ldots,$$

on multiplying the two series and grouping terms of the same degree in z_1 and z_2 together, this being possible because both the series are absolutely convergent.

It follows that

$$E(z_1)\times E(z_2)=1+(z_1+z_2)+\frac{1}{2!}(z_1+z_2)^2$$
$$+\ldots+\frac{(z_1+z_2)^n}{n!}+\ldots,$$
$$=E(z_1+z_2).$$

Further, since the series for $E(z_1)$ and $E(z_2)$ are absolutely convergent, by a theorem in Algebra, the series for their product, viz. $E(z_1+z_2)$ is also absolutely convergent.

5·12. The exponential functions for real values. On account of the property established in § 5·11, it would be appropriate to identify $E(z)$ as e^z. We have thus

$$e^z=1+z+\frac{z^2}{2!}+\ldots+\frac{z^n}{n!}+\ldots$$

for all values of z, real or complex.

The student should note that, when x is real, e^x has the usual significance, *viz.* the quantity

$$e=1+\frac{1}{1!}+\frac{1}{2!}+\dots$$

raised to the power x. In other words,

$$\left(1+\frac{1}{1!}+\frac{1}{2!}+\dots\right)^x=1+x+\frac{x^2}{2!}+\dots.$$

But e^z has no such significance and is a symbol for the series (2) of § 5·1.

Sometimes it is convenient to write e^z as $\exp(z)$.

The result of § 5·11 may thus be written as

$$e^{z_1}\,e^{z_2}=e^{z_1+z_2},$$

or $$\exp(z_1)\times\exp(z_2)=\exp(z_1+z_2).$$

COROLLARY. $\exp(z_1)\div\exp(z_2)=\exp(z_1-z_2)$.

5·13. An important proposition.

To prove that $\{exp(z)\}^n=exp(nz)$.

We have, by the help of § 5·11,

$$\begin{aligned}\exp(z_1)\times\exp(z_2)\times\dots\times\exp(z_n)&\\ =\exp(z_1+z_2)\times\exp(z_3)\times\dots\times\exp(z_n)&\\ =\exp(z_1+z_2+z_3)\times\exp(z_4)\times\dots&\\ \times\exp(z_n)&\\ =\exp(z_1+z_2+\dots+z_n).&\end{aligned}$$

If we put $z_1=z_2=\dots=z_n=z$, we have the desired result.

NOTE 1. The above result may be easily extended to the case when n is any real number. It should, however, be noted that when n is rational, $\exp(nz)$ will be one of the values of $\{\exp(z)\}^n$.

NOTE 2. From the above we have that, whether θ be real or complex,

$$(e^{i\theta})^n=e^{in\theta},$$

or $$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta,$$

which shows that De Moivre's theorem is true whether θ be real or complex.

CIRCULAR FUNCTIONS OF COMPLEX QUANTITIES

5·2. Definitions. We have seen in § 4·3 and § 4·31 that for real values of x

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-)^n \frac{x^{2n+1}}{(2n+1)!} + \dots,$$

and $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + (-)^n \frac{x^{2n}}{(2n)!} + \dots.$

We now extend these for the complex quantity $z = x + iy$, where x and y are real, and define $\sin z$ and $\cos z$ as

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots + (-)^n \frac{z^{2n+1}}{(2n+1)!} + \dots$$

and $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots + (-)^n \frac{z^{2n}}{(2n)!} + \dots.$

The other circular functions for a complex quantity are defined in the same way as those for a real quantity. Thus

$$\tan z = \frac{\sin z}{\cos z}, \ \cot z = \frac{\cos z}{\sin z},$$

$$\sec z = \frac{1}{\cos z}, \ \operatorname{cosec} z = \frac{1}{\sin z}.$$

5·3. Further properties of exponential functions. We are now in a position to deduce some further important properties of $\exp(z)$.

(i) Euler's Theorem:

$$e^{i\theta}=\cos\theta+i\sin\theta,$$

whether θ be real or complex.

We have, from the definitions of $\cos\theta$ and $\sin\theta$,

$$\cos\theta+i\sin\theta=\left(1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-\ldots\right)+i\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\ldots\right)$$

$$=1+i\theta+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\ldots,$$

on arranging the terms in powers of $i\theta$,

$$=e^{i\theta}.$$

COROLLARY 1. $e^{-i\theta}=\cos\theta-i\sin\theta$.

COROLLARY 2. $e^{2n\pi i}=1$.

NOTE. Euler's theorem is of great importance as it enables us to write down the real and imaginary parts of an exponential function separately. As for example, e^z, where $z=x+iy$, can be written as

$$e^z=e^{x+iy}$$

$$=e^x\,.\,e^{iy}$$

$$=e^x(\cos y+i\sin y), \text{ using Euler's theorem,}$$

$$=e^x\cos y+i\,e^x\sin y,$$

where $e^x\cos y$ is the real part of e^z, and $i\,e^x\sin y$ the imaginary part of e^z.

(ii) $\cos\theta=\frac{1}{2}(e^{i\theta}+e^{-i\theta})$ *and* $\sin\theta=\dfrac{1}{2i}(e^{i\theta}-e^{-i\theta})$.

These follow at once on adding and subtracting the following formulae of (i);

$$e^{i\theta}=\cos\theta+i\sin\theta \text{ and } e^{-i\theta}=\cos\theta-i\sin\theta.$$

COROLLARY 1. $\tan\theta=\dfrac{e^{i\theta}-e^{-i\theta}}{i(e^{i\theta}+e^{-i\theta})}$,

and $\cot\theta=\dfrac{i(e^{i\theta}+e^{-i\theta})}{(e^{i\theta}-e^{-i\theta})}$.

COROLLARY 2. The period of complex circular functions is 2π.

(iii) *$Exp(z)$ is a periodic function of period $2\pi i$.*

If $z=x+iy$, then

$$\exp(z)=\exp(x+iy)=\exp(x)\exp(iy)$$
$$=e^x(\cos y+i\sin y).$$

But $\cos y$ and $\sin y$ remain unchanged when y is increased by $2n\pi$, where n is any positive or negative integer. Therefore,

$$\exp(z)=e^x[\cos(y+2n\pi)+i\sin(y+2n\pi)]$$
$$=e^x \,.\, e^{(y+2n\pi)i}$$
$$=e^{x+iy+2n\pi i}$$
$$=\exp(z+2n\pi i).$$

5·4. Addition theorem for circular functions. We are now in a position to show that

$$\cos(\theta\pm\phi)=\cos\theta\cos\phi\mp\sin\theta\sin\phi$$

and $$\sin(\theta\pm\phi)=\sin\theta\cos\phi\pm\cos\theta\sin\phi$$

where θ and ϕ have any value, real or complex.

For example, from § 5·11, we have

$$e^{i(\theta+\phi)}=e^{i\theta}e^{i\phi}.$$

If we use Euler's theorem, this gives that

$$\cos(\theta+\phi)+i\sin(\theta+\phi)$$
$$=(\cos\theta+i\sin\theta)(\cos\phi+i\sin\phi)$$
$$=\cos\theta\cos\phi-\sin\theta\sin\phi$$
$$+i(\sin\theta\cos\phi+\cos\theta\sin\phi). \qquad (1)$$

Similarly, starting with

$$e^{-i(\theta+\phi)}=e^{-i\theta}\,e^{-i\phi}$$

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we have

$$\cos(\theta+\phi)-i\sin(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi$$
$$-i(\sin\theta\cos\phi+\cos\theta\sin\phi). \quad (2)$$

From (1) and (2), by adding and subtracting, we obtain

$$\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi,$$

and $$\sin(\theta+\phi)=\sin\theta\cos\phi+\cos\theta\sin\phi,$$

for θ and ϕ, real or complex.

If, however, we start with the identities

$$e^{i(\theta-\phi)}=e^{i\theta}\,e^{-i\phi}$$

and $$e^{-i(\theta-\phi)}=e^{-i\theta}\,e^{i\phi}$$

and proceed as before, we shall get

$$\cos(\theta-\phi)=\cos\theta\cos\phi+\sin\theta\sin\phi,$$

and $$\sin(\theta-\phi)=\sin\theta\cos\phi-\cos\theta\sin\phi.$$

5·41. Generality of trigonometrical formulae. In fact, all the trigonometrical formulae which are valid for real values and which are obtained by addition and subtraction can be shown to be true for complex quantities as well.

Thus, when $z=x+iy$, $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$,

$$\sin 2z=2\sin z\cos z,$$
$$\cos 2z=\cos^2 z-\sin^2 z,$$
$$\sin 3z=3\sin z-4\sin^3 z,$$
$$\cos 3z=4\cos^3 z-3\cos z,$$
$$\tan(z_1+z_2)=\frac{\tan z_1+\tan z_2}{1-\tan z_1\tan z_2}$$
$$\cot(z_1+z_2)=\frac{\cot z_1\cot z_2-1}{\cot z_1+\cot z_2},$$

and so on.

Ex. If $z=x+iy$, where x and y are real, find the modulus and amplitude of e^{z^2} and the real and imaginary parts of

$$\frac{\cos z}{z+1}.$$ [*Luck.*, 1968]

If $z=x+iy$ then $z^2=x^2-y^2+2ixy$, so that

$$e^{z^2}=e^{x^2-y^2+2ixy}$$
$$=e^{x^2-y^2}(\cos 2xy+i\sin 2xy).$$

Hence $$\text{mod}\,(e^{z^2})=e^{x^2-y^2},$$

and $$\text{amp}(e^{z^2})=2xy.$$

To find the real and imaginary parts of $\cos z/(z+1)$, we first find the real and imaginary parts of $\cos z$ and $1/(z+1)$ separately.

Now $$\cos z=\cos(x+iy)$$
$$=\cos x\cos iy-\sin x\sin iy$$
$$=\cos x\left(\frac{e^{-y}+e^{y}}{2}\right)-\sin x\left(\frac{e^{-y}-e^{y}}{2i}\right)$$
$$=\tfrac{1}{2}(e^{y}+e^{-y})\cos x-\tfrac{1}{2}i(e^{y}-e^{-y})\sin x$$

and $$\frac{1}{1+z}=\frac{1}{1+x+iy}=\frac{1+x-iy}{(1+x)^2+y^2}.$$

Hence

$$\frac{\cos z}{z+1}=[\tfrac{1}{2}(e^{y}+e^{-y})\cos x-\tfrac{1}{2}i(e^{y}-e^{-y})\sin x]\times\left[\frac{1+x-iy}{(1+x)^2+y^2}\right]$$

$$=[\tfrac{1}{2}(1+x)(e^{y}+e^{-y})\cos x-\tfrac{1}{2}(e^{y}-e^{-y})y\sin x]/\{(1+x)^2+y^2\}$$
$$-i[\tfrac{1}{2}(e^{y}+e^{-y})y\cos x+\tfrac{1}{2}(1+x)(e^{y}-e^{-y})\sin x]/\{(1+x)^2+y^2\}.$$

EXAMPLES

1. Show that

$$\exp\left(\pm i\frac{\pi}{2}\right)=\pm i.$$

2. Apply the exponential values of the sine and cosine to show that

(i) $\cos(-z)=\cos z$ and $\sin(-z)=-\sin z$,

(ii) $\sin^2 z + \cos^2 z = 1$,

(iii) $\cos 2z = 2\cos^2 z - 1 = 1 - 2\sin^2 z$,

(iv) $\dfrac{\sin 2z}{1-\cos 2z} = \cot z$,

and (v) $\cos z_1 - \cos z_2 = 2 \sin \frac{1}{2}(z_1+z_2) \sin \frac{1}{2}(z_2 - z_1)$.

3. Separate the real and imaginary parts of the following :

(i) $\exp(e^{i\theta})$,
(ii) $\exp\{(x+iy)(\alpha+i\beta)\}$,
(iii) $\sin(\theta i)$,
(iv) $\cos(\theta - i\phi)$,
(v) $\sin(\theta i)/(x+iy)$,
(vi) $\exp(\sin i\theta)$,
(vii) $\tan(\theta i) \cdot e^{\phi i}$,
(viii) $\sec(x+iy)$. [*Luck.*, 1968]

4. Prove that

(i) $\{\sin(\alpha+\theta) - e^{\alpha i} \sin \theta\}^n = \sin^n \alpha \cdot e^{-n\theta i}$,

(ii) $\{\sin(\alpha-\theta) + e^{\pm \alpha i} \sin \theta\}^n$
$= \sin^{n-1} \alpha \{\sin(\alpha - n\theta) + e^{\pm \alpha i} \sin n\theta\}$.

5. If α, β be the imaginary cube roots of unity, prove that

$$\alpha e^{\alpha x} + \beta e^{\beta x} = -e^{-x/2}\left[\sqrt{3} \sin \frac{\sqrt{3}}{2}x + \cos \frac{\sqrt{3}}{2}x\right].$$

[*Bombay*, 1947]

6. If $\exp\left(\dfrac{x-a+iy}{x+a+iy}\right) = X + iY$, find X and Y.

LOGARITHM OF A COMPLEX QUANTITY

5·5. Definition. If $e^z = w$, where $z \equiv x+iy$ and $w \equiv u+iv$ are complex quantities, then z is called the logarithm (Napierian) of w and is written as

$$z = \log_e w.$$

Since $$e^{2n\pi i}=1,$$
we have also $$e^{z+2n\pi i}=w,$$
giving that $$\log_e w=z+2n\pi i,$$
where n is zero or any integer, positive or negative.

This shows that the logarithm of a complex quantity has an infinite number of values and is hence a many-valued function. The principal value of the logarithm of w is obtained by taking $n=0$.

Since the logarithm of a complex quantity is defined in the same way as the logarithm of a real quantity, it is clear that the results

$$\log uv=\log u+\log v,$$
$$\log (u/v)=\log u-\log v,$$
$$\log u^v=v\log u,$$

will be true when u and v are any complex quantities.

5·51. Logarithms of a real number. Let us now suppose that $e^x=a$, where x and a are both real. We can write this as

$$e^x=e^{x+2n\pi i}=a,$$

n being any integer. If follows that

$$x+2n\pi i=\log a,$$

showing that a real number has one real logarithm (corresponding to $n=0$) and an infinite number of imaginary logarithms.

It would be convenient to denote the general value of the logarithm of w by the notation Log w and the principal value by log w. We then have

$$\text{Log } w=2n\pi i+\log w.$$

5·52. The principal value of a logarithm.

Let us now suppose that

$$z=x+iy=r(\cos\theta+i\sin\theta)=re^{\theta i},$$

where $r=\sqrt{(x^2+y^2)}$ and $\theta=\tan^{-1}(y/x)$.

Then

$$\begin{aligned}\text{Log}\, z &= 2n\pi i+\log z=2n\pi i+\log re^{\theta i}\\ &=\log r+i(\theta+2n\pi).\end{aligned}$$

The principal value of the logarithm of z (obtained by taking $n=0$) is

$$\log z\equiv\log(x+iy)=\log\sqrt{(x^2+y^2)}+i\tan^{-1}(y/x).$$

NOTE. It is to be noted that the formula

$$\log(x+iy)=\log\sqrt{x^2+y^2}+i\tan^{-1}(y/x)$$

expresses the real and imaginary parts of $\log(x+iy)$ separately. The student is advised to remember this formula and make use of it directly whenever he has to separate the real and imaginary parts of a logarithmic function.

Ex. 1. Find the general value of $\log\sqrt{i}$.

$$\begin{aligned}\text{Log}\sqrt{i} &= 2n\pi i+\log\sqrt{i}\\ &=2n\pi i+\tfrac{1}{2}\log i\\ &=2n\pi i+\tfrac{1}{2}\log(\cos\tfrac{1}{2}\pi+i\sin\tfrac{1}{2}\pi)\\ &=2n\pi i+\tfrac{1}{2}\log e^{(1/2)\pi i}\\ &=2n\pi i+\tfrac{1}{2}\cdot\tfrac{1}{2}\pi i\\ &=(2n\pi+\tfrac{1}{4}\pi)i.\end{aligned}$$

Ex. 2. Prove that

$$\tan\left\{i\log\frac{a-ib}{a+ib}\right\}=\frac{2ab}{a^2-b^2}.$$

[*Lucknow*, 1966]

We have

$$i\log\frac{a-ib}{a+ib}=i\Big[\log(a-ib)-\log(a+ib)\Big]$$

$$=i\left[\left\{\tfrac{1}{2}\log(a^2+b^2)-i\tan^{-1}\frac{b}{a}\right\}\right.$$

$$\left.-\left\{\tfrac{1}{2}\log(a^2+b^2)+i\tan^{-1}\frac{b}{a}\right\}\right]$$

$$=i\left(-2i\tan^{-1}\frac{b}{a}\right)$$

$$=2\tan^{-1}\frac{b}{a}=\tan^{-1}\frac{2(b/a)}{1-(b/a)^2}$$

$$=\tan^{-1}\frac{2ab}{a^2-b^2}.$$

Therefore $\tan\left\{i\log\dfrac{a-ib}{a+ib}\right\}=\dfrac{2ab}{a^2-b^2}.$

Ex. 3. If $(a_1+ib_1)(a_2+ib_2)\dots(a_n+ib_n)=A+iB$, prove that

$$\tan^{-1}\frac{b_1}{a_1}+\tan^{-1}\frac{b_2}{a_2}+\dots+\tan^{-1}\frac{b_n}{a_n}=\tan^{-1}\frac{B}{A},$$

and $(a_1^2+b_1^2)(a_2^2+b_2^2)\dots(a_n^2+b_n^2)=A^2+B^2.$

[*Lucknow*, 1963]

Since

$$(a_1+ib_1)(a_2+ib_2)\dots(a_n+ib_n)=A+iB,$$

therefore, on taking the principal values of the logarithms of the two sides,

$$\log(a_1+ib_1)+\log(a_2+ib_2)+\dots+\log(a_n+ib_n)=\log(A+iB)$$

or $\quad[\tfrac{1}{2}\log(a_1^2+b_1^2)+i\tan^{-1}(b_1/a_1)]+[\tfrac{1}{2}\log(a_2^2+b_2^2)$
$+i\tan^{-1}(b_2/a_2)]+\dots+[\tfrac{1}{2}\log(a_n^2+b_n^2)+i\tan^{-1}(b_n/a_n)]$
$=\tfrac{1}{2}\log(A^2+B^2)+i\tan^{-1}(B/A).$

Hence equating real and imaginary parts, we get the required results.

Examples

1. Prove that

(i) $\log(-1)=\pi i$, [*Luck.*, '65]

and (ii) $\text{Log}(-1)=i(2n+1)\pi$.

2. Find the general value of $\log(-3)$. [*Lucknow*, '66]

3. Show that $\text{Log } i = i(2n+\frac{1}{2})\pi$. [*Lucknow*, 1959]

4. Prove that $\log(\cos\theta + i\sin\theta) = \theta i$.

5. Show that

$$\text{Log }(1+i) = \tfrac{1}{2}\log 2 + (2n+\tfrac{1}{4})\pi i.$$

6. Prove that the general value of

$$\log(1+\cos 2\theta + i\sin 2\theta) = \log(2\cos\theta) + i(\theta + 2k\pi),$$

where k is any integer. [*Mysore*, 1941]

7. If $A+iB = \log(p+iq)$, show that

$$A = \tfrac{1}{2}\log(p^2+q^2) \quad \text{and} \quad B = \tan^{-1}(q/p).$$

8. Show that

$$\begin{aligned}\log(a+ib-x)(a-ib+x) &= \tfrac{1}{2}\log[\{(a-x)^2+b^2\}\{(a+x)^2+b^2\}] \\ &\quad + i(\tan^{-1}\{b/(a-x)\} - \tan^{-1}\{b/(a+x)\}].\end{aligned}$$

9. Find the general and principal values of

$$\log(-1+i) - \log(-1-i).$$

10. Show that

$$\log\log(x+iy) = \tfrac{1}{2}\log(p^2+q^2) + i\tan^{-1}(q/p),$$

where $\quad 2p = \log(x^2+y^2) \quad$ and $\quad q = \tan^{-1}(y/x)$.

11. Show that one of the values of

$$\log\{(1+i)(1+i\sqrt{3})/(\sqrt{3}+i)\}$$

is

$$\tfrac{1}{2}\log 2 + i\tfrac{5}{12}\pi.$$ [*Dacca*, 1933]

12. Prove that

$$\text{Log }\frac{a-b+i(a+b)}{a+b+i(a-b)} = i\left\{2n\pi + \tan^{-1}\frac{2ab}{a^2-b^2}\right\}.$$ [*Dacca*, 1951]

13. Express $\log_a(1+i)^{1-i}$ is the form $A+iB$. [*Luck.*, '68]

THE GENERAL EXPONENTIAL FUNCTION

5·6. Definition. If a be any number, real or complex, the symbol a^z is defined to mean $\exp(z \text{ Log } a)$, where $\text{Log } a$ has any of its infinite number of values. When $\text{Log } a$ has its principal value $\log a$, we shall call $\exp(z\log a)$ the principal value of a^z.

Since

$$\exp(z\,\mathrm{Log}\,a)=1+\frac{z\,\mathrm{Log}\,a}{1\,!}+\frac{(z\,\mathrm{Log}\,a)^2}{2\,!}+\ldots,$$

we have the general exponential function

$$a^z=1+\frac{z\,\mathrm{Log}\,a}{1\,!}+\frac{(z\,\mathrm{Log}\,a)^2}{2\,!}+\ldots.$$

The principal value of a^z is given by

$$a^z=1+\frac{z\log a}{1\,!}+\frac{(z\log a)^2}{2\,!}+\ldots.$$

5·61. The exponential function is many-valued. The many-valued nature of the function a^z, as defined above, may also be demonstrated as follows :

Let $a=\alpha+i\beta=r(\cos\theta+i\sin\theta)=re^{i\theta}$

and $z=x+iy$.

Now

$$\begin{aligned}\mathrm{Log}\,a&=2n\pi i+\log a\\&=2n\pi i+\log re^{i\theta}\\&=\log r+i(\theta+2n\pi).\end{aligned}$$

Hence

$$\begin{aligned}a^z&=\exp\,[z\,\mathrm{Log}\,a]\\&=\exp\,[(x+iy)\,\{\log r+i(\theta+2n\pi)\}]\\&=\exp\,[\{x\log r-y(\theta+2n\pi)\}\\&\qquad+i\{y\log r+x(\theta+2n\pi)\}].\end{aligned}$$

The principal value is obtained by taking $n=0$.

5·62. The real and imaginary parts of the exponential function. We can now reduce a^z to the form $A+iB$.

As given above

$$a^z = \exp [\{x \log r - y(\theta+2n\pi)\} + i\{y \log r + x(\theta+2n\pi)\}]$$

$$= \exp \{x \log r - y(\theta+2n\pi)\} \times [\cos \{y \log r + x(\theta+2n\pi)\} + i \sin \{y \log r + x(\theta+2n\pi)\}],$$

on using Euler's theorem.

Hence $A = \exp \{x \log r - y(\theta+2n\pi)\} \cos \{y \log r + x(\theta+2n\pi)\}$,

and $B = \exp \{x \log r - y(\theta+2n\pi)\} \sin \{y \log r + x(\theta+2n\pi)\}$.

Ex. 1. Show that

$$i^i = e^{-(4n+1)\pi/2}. \qquad [\textit{Lucknow}, 1946]$$

By definition.

$$i^i = \exp (i \operatorname{Log} i) = \exp \{i(2n\pi i + \log i)\}$$
$$= \exp \{i (2n\pi i + \log e^{\pi i/2})\}$$
$$= e^{i(2n\pi i + \pi i/2)}$$
$$= e^{-(4n+1)\pi/2}.$$

NOTE. By putting $n = 0, 1, 2, \ldots$, we see that the values of i form a geometrical progression whose common ratio is $e^{-2\pi}$.

Ex. 2. If $u = \log_e \tan (\frac{1}{4}\pi + \frac{1}{2}x) = x + a_3x^3 + a_5x^5 + \ldots$, prove that

$$x = u - a_3u^3 + a_5u^5 - \ldots . \qquad [\textit{Alld.}, \text{'52}]$$

Since $u = \log_e \tan (\frac{1}{4}\pi + \frac{1}{2}x)$,

therefore $e^u = \tan (\frac{1}{4}\pi + \frac{1}{2}x)$

$$= \frac{1 + \tan \frac{1}{2}x}{1 - \tan \frac{1}{2}x}.$$

Applying Componendo and Dividendo, we have

$$\tan (\tfrac{1}{2}x) = (e^u - 1)/(e^u + 1)$$
$$= -\{e^{(ui/2)i} - e^{-(ui/2)i}\}/\{e^{(ui/2)i} + e^{-(ui/2)i}\}$$

$$= -i \tan\left(\tfrac{1}{2}ui\right),$$

or $$\frac{i \sin\left(\frac{1}{2}x\right)}{\cos\left(\frac{1}{2}x\right)} = \tan\left(\tfrac{1}{2}ui\right).$$

Applying Componendo and Dividendo again,

$$\frac{\cos\left(\frac{1}{2}x\right) + i \sin\left(\frac{1}{2}x\right)}{\cos\left(\frac{1}{2}x\right) - i \sin\left(\frac{1}{2}x\right)} = \frac{1 + \tan\left(\frac{1}{2}ui\right)}{1 - \tan\left(\frac{1}{2}ui\right)}$$

or $$e^{xi} = \tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}ui\right).$$

Therefore, $xi = \log_e \tan\left(\frac{1}{4}\pi + \frac{1}{2}ui\right)$,

or $$x = \frac{1}{i} \log_e \tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}ui\right)$$

$$= \frac{1}{i}\{ui + a_3(ui)^3 + a_5(ui)^5 + \ldots\}, \text{ by hypothesis}$$

$$= u - a_3u^3 + a_5u^5 - \ldots .$$

Examples

1. Prove that

$$x^i = e^{-2n\pi}\{\cos(\log x) + i \sin(\log x)\}$$

and $$i^x = \cos\left(2n + \tfrac{1}{2}\right)\pi x + i \sin\left(2n + \tfrac{1}{2}\right)\pi x.$$

2. Express the following in the form $A + iB$, where A and B are real :

(i) $i^{\alpha + i\beta}$, (ii) $(i^i)^i$, (iii) i^{i^i}.

3. Find the principal value of $(3 + 4i)^i$. [*Mysore*, 1945]

4. Show that

$$(\sin x + i \cos x)^i = e^{x - 2n\pi - \pi/2}.$$

5. Show that

$$\sin(\log i^i) = -1.$$ [*Lucknow*, '57]

6. Show that

$$(-i)^{-i} = e^{(4n-1)\pi/2}.$$

7. If i^{i^i}...ad inf. $= A + iB$, principal values only being considered, prove that

$$\tan \tfrac{1}{2}\pi A = B/A \quad \text{and} \quad A^2 + B^2 = e^{-\pi B}.$$

[*Jammu and Kashmir*, 1953]

8. Show that the real part of the principal value of $i^{\log(1+i)}$ is

$$e^{-\pi^2/8}\cos\left(\tfrac{1}{4}\pi\log 2\right).$$

9. Show that the ratio of the principal values of $(1+i)^{1-i}$ and $(1-i)^{1+i}$ is

$$\sin(\log 2)+i\cos(\log 2).$$

10. Show that the principal value of $(x+iy)^{a+ib}$ is wholly real or wholly imaginary according as

$$\tfrac{1}{2}b\log(x^2+y^2)+a\tan^{-1}(y/x)$$

is an even or odd multiple of $\frac{1}{2}\pi$. [*Allahabad*, 1941]

11. If $\dfrac{(1+i)^{p+qi}}{(1-i)^{p-qi}}=x+iy$, prove that one of the values of $\tan^{-1}(y/x)$ is $\frac{1}{2}p\pi+q\log_e 2$. [*Lucknow*, 1961]

12. Show that

$$\operatorname{Log}_i i=\frac{4m+1}{4n+1},$$

where m and n are integers. [*Andhra*, 1940]

THE INVERSE CIRCULAR FUNCTIONS OF COMPLEX QUANTITIES

5·7. Definition. We shall first consider $\sin^{-1}w$ where $w\equiv u+iv$, u and v being real. Its value $z\equiv x+iy$ is defined by

$$\sin(x+iy)=u+iv.$$

Since $\sin(x+iy)=\sin\{n\pi+(-)^n(x+iy)\}$,
we learn, by the above definition that the general value of the inverse sine of $u+iv$ is

$$\pi n+(-)^n(x+iy).$$

This shows that $\sin^{-1}(u+iv)$ is a many-valued function. The principal value is that in which the real part lies between $-\pi/2$ and $\pi/2$.

Similarly, if $\cos(x+iy)=u+iv$, then $x+iy$ is said to be an inverse cosine of $u+iv$.

Also since $\cos\{2n\pi\pm(x+iy)\}=u+iv$, therefore $2n\pi\pm(x+iy)$ is the general value of $\cos^{-1}(u+iv)$, showing the many-valued nature of the function, and the principal value is that in which the real part lies between 0 and π.

Again if $\tan(x+iy)=u+iv$,

then $$u+iv=\tan(n\pi+x+iy),$$

giving $$\tan^{-1}(u+iv)=n\pi+(x+iy).$$

The principal value of $\tan^{-1}(u+iv)$ is the one in which the real part lies between $-\pi/2$ and $\pi/2$.

If we choose to write the general value of an inverse circular function of a complex quantity by writing the first letter as capital one and the principal value in the ordinary way, the above relation and the corresponding relations of the remaining circular functions may be written as follows:

$$\left\{\begin{aligned}
\text{Sin}^{-1}(u+iv)&=n\pi+(-)^n\sin^{-1}(u+iv),\\
\text{Cos}^{-1}(u+iv)&=2n\pi\pm\cos^{-1}(u+iv),\\
\text{Tan}^{-1}(u+iv)&=n\pi+\tan^{-1}(u+iv),\\
\text{Cot}^{-1}(u+iv)&=n\pi+\cot^{-1}(u+iv),\\
\text{Sec}^{-1}(u+iv)&=2n\pi\pm\sec^{-1}(u+iv),
\end{aligned}\right.$$

and $$\text{Cosec}^{-1}(u+iv)=n\pi+(-)^n\operatorname{cosec}^{-1}(u+iv).$$

Ex. Prove that $\sin^{-1}(ix)=n\pi+i(-1)^n\log\{\sqrt{(1+x^2)}+x\}$.

Let $$\sin^{-1}(ix)=z.$$

Then $$\sin z=ix,$$

or $$\frac{e^{zi}-e^{-zi}}{2i}=ix,$$

$\therefore$ $$e^{zi}-e^{-zi}=-2x.$$

Multiplying by e^{zi} and transposing,

$$e^{2zi}+2xe^{zi}-1=0.$$

Solving this equation for e^{zi}, we have

$$e^{zi}=-x\pm\sqrt{(x^2+1)}.$$

Therefore

$$zi=2m\pi i+\log\{\sqrt{(x^2+1)}-x\}$$

or

$$=2m\pi i+\log(-1)+\log\{\sqrt{(x^2+1)}+x\}.$$

But $$\sqrt{(x^2+1)}-x=\frac{(x^2+1)-x^2}{\sqrt{(x^2+1)}+x}=\frac{1}{\sqrt{(x^2+1)}+x},$$

and $$\log(-1)=\pi i.$$

Therefore

$$zi=2m\pi i-\log\{\sqrt{(x^2+1)}+x\}$$

or

$$=(2m+1)\pi i+\log\{\sqrt{(x^2+1)}+x\}.$$

$\therefore$ $$z=2m\pi+i\log\{\sqrt{(x^2+1)}+x\}$$

or

$$=(2m+1)\pi-i\log\{\sqrt{(x^2+1)}+x\}.$$

Both these values of z are included in the expression

$$n\pi+i(-1)^n\log\{\sqrt{(x^2+1)}+x\}.$$

Hence

$$\sin^{-1}(ix)=n\pi+i(-1)^n\log\{\sqrt{(x^2+1)}+x\}.$$

Alternative Method. Let $\sin^{-1}(ix)=z$. Then

$$\sin z=ix \text{ and } \cos z=\sqrt{(x^2+1)}.$$

$\therefore$ $$e^{zi}=\cos z+i\sin z=\sqrt{(x^2+1)}-x$$

$$=\frac{1}{\sqrt{(x^2+1)}+x}.$$

$\therefore$ $$zi=-\log\{\sqrt{(x^2+1)}+x\}.$$

Therefore $z=\sin^{-1}(ix)=i\log\{\sqrt{(x^2+1)}+x\}$.

Hence $\text{Sin}^{-1}(xi)=n\pi+i(-1)^n\log\{\sqrt{(x^2+1)}+x\}$.

EXAMPLES ON CHAPTER V

1. By using the exponential value of the cosine, prove the identity

$$\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1,$$

connecting the angles A, B, C of any triangle. [*Luck.*, 1968]

2. Prove that

$$\sin(\alpha+n\theta)-e^{\alpha i}\sin n\theta=e^{-n\theta i}\sin\alpha.$$

3. Prove that the principal value of

$$(a+ia\tan\phi)^{\log(a\sec\phi)-i\phi}$$

is wholly real and find it. [*Lucknow*, '57]

4. Show that

$$\log\frac{1}{1-e^{\theta i}}=\log(\tfrac{1}{2}\operatorname{cosec}\tfrac{1}{2}\theta)+\tfrac{1}{2}(\pi-\theta)i.$$

5. If $a+ib=q^{x+iy}$, prove that

$$\frac{y}{x}=\frac{2\tan^{-1}(b/a)}{\log(a^2+b^2)}.$$ [*Agra*, 1959]

6. If $\tan\log(\alpha+i\beta)=x-iy$, where $x^2+y^2\neq 1$, prove that

$$\tan\log(\alpha^2+\beta^2)=2x/(1-x^2-y^2).$$ [*Calcutta*, 1946]

[HINT. Since

$$\log(\alpha+i\beta)=\tan^{-1}(x-iy),$$

and likewise $\log(\alpha-i\beta)=\tan^{-1}(x+iy)$,

therefore $\tan\{\log(\alpha+i\beta)+\log(\alpha-i\beta)\}=2x/(1-x^2-y^2)$].

7. Show that the principal value of

$$\frac{(a+ib)^{p+iq}}{(a-ib)^{p-iq}}$$

is $\cos 2(p\alpha+q\log r)+i\sin 2(p\alpha+q\log r)$, where

$$r=\sqrt{(a^2+b^2)}\quad\text{and}\quad \alpha=\tan^{-1}(b/a).$$

[*Lucknow*, 1952]

8. If $\log\log(x+iy)=p+iq$, show that

$$y=x\tan\theta,$$

where $2\theta=\tan q\,.\,\log(x^2+y^2)$. [*Annam.*, 1939]

9. Show that

$$\tan^{-1}\left(i\frac{x-a}{x+a}\right)=-\frac{i}{2}\log\frac{a}{x}.$$ [*Lucknow*, '64]

10. Prove that the general value of $(1+i\tan\alpha)^{-i}$ is

$$e^{\alpha+2m\pi}[\cos(\log\cos\alpha)+i\sin(\log\cos\alpha)],$$

where m is any integer. [*Baroda*, 1954]

11. If all the values of $(1+i\tan\alpha)^{1+i\tan\beta}$ are real, prove that one of them is

$$(\sec\alpha)^{\sec^2\beta}.$$ [*Lucknow*, '66]

12. Express the logarithms of $x+iy$ to the base $\alpha+i\beta$, in the form $A+iB$.

13. If $\sin^{-1}(x+iy)=\tan^{-1}(\xi+i\eta)$, show that

$$[(x-1)^2+y^2][(x+1)^2+y^2]=\frac{(x^2+y^2)^2}{(\xi^2+\eta^2)^2}.$$

[*Lucknow*, '56]

[HINT. $\sin^{-1}(x+iy)=\tan^{-1}\dfrac{x+iy}{\sqrt{[1-(x+iy)^2]}}=\tan^{-1}(\xi+i\eta)$, (1)

and $\sin^{-1}(x-iy)=\tan^{-1}\dfrac{x-iy}{\sqrt{[1-(x-iy)^2]}}=\tan^{-1}(\xi-i\eta)$. (2)

Multiply (1) and (2).]

14. If $$\left(\frac{a+x+iy}{a-x-iy}\right)^{\lambda+i\mu}=P+iQ,$$

prove that one of the values of $\tan^{-1}(Q/P)$ is

$$\lambda\tan^{-1}\{2ay/(a^2-x^2-y^2)\}+\tfrac{1}{2}\mu\log[\{(a+x)^2+y^2\}/\{(a-x)^2+y^2\}].$$

[*Gauhati*, 1953]

15. If $a^{\alpha+i\beta}=(x+iy)^{p+iq}$, principal values only being considered, prove that

$$\alpha=\tfrac{1}{2}p\log_a(x^2+y^2)-q\tan^{-1}(y/x)\,.\log_a\epsilon,$$

and that $$\log_a(x^2+y^2)=\frac{2(\alpha p+\beta q)}{p^2+q^2}.$$

[*Banaras*, '62]

16. Prove that

$$\sin^{-1}(\operatorname{cosec}\theta)=\{2n+(-1)^n\}\pi/2$$
$$+i(-1)^n\log\cot(\theta/2).$$

17. If $x<\frac{\pi}{2}$, show that $i\cos^{-1}(\sin x+\cos x)$ has two real values.

18. Show that

$$\lim_{n\to1}\frac{1+(-1)^n}{1-n}=\pi i.$$

CHAPTER VI

HYPERBOLIC FUNCTIONS

6·1. Definitions. The expressions $\frac{1}{2}(e^{\theta}+e^{-\theta})$, and $\frac{1}{2}(e^{\theta}-e^{-\theta})$, where the exponentials have their principal values, are called hyperbolic cosine and hyperbolic sine of θ and are denoted by $\cosh\theta$ and $\sinh\theta$ respectively. It follows on expanding the exponential functions that

$$\cosh\theta=\tfrac{1}{2}(e^{\theta}+e^{-\theta})=1+\frac{\theta^2}{2!}+\frac{\theta^4}{4!}+\frac{\theta^6}{6!}+\dots,$$

and $$\sinh\theta=\tfrac{1}{2}(e^{\theta}-e^{-\theta})=\theta+\frac{\theta^3}{3!}+\frac{\theta^5}{5!}+\frac{\theta^7}{7!}+\dots.$$

Just as the ordinary tangent, cotangent, secant and cosecant of an angle can be obtained from the ordinary sine and cosine of the angle, so the hyperbolic tangent, cotangent, secant and cosecant are defined in terms of the hyperbolic sine and cosine. Thus

$$\tanh\theta=\frac{\sinh\theta}{\cosh\theta}=\frac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}},$$

$$\coth\theta=\frac{\cosh\theta}{\sinh\theta}=\frac{e^{\theta}+e^{-\theta}}{e^{\theta}-e^{-\theta}},$$

$$\operatorname{sech}\theta=\frac{1}{\cosh\theta}=\frac{2}{e^{\theta}+e^{-\theta}}$$

and $$\operatorname{cosech}\theta=\frac{1}{\sinh\theta}=\frac{2}{e^{\theta}-e^{-\theta}}.$$

The above hyperbolic functions are expressed in terms of the corresponding circular functions by the equations

$$\sinh\theta = -i\sin i\theta,\quad \cosh\theta=\cos i\theta,\quad \tanh\theta=-i\tan i\theta,$$

$$\coth\theta = i\cot i\theta,\quad \operatorname{sech}\theta=\sec i\theta,$$

and $\operatorname{cosech}\theta = i\operatorname{cosec} i\theta.$

And conversely,

$$\sin\theta = -i\sinh i\theta,\quad \cos\theta=\cosh i\theta,\quad \tan\theta=-i\tanh i\theta,$$

$$\cot\theta = i\coth i\theta,\quad \sec\theta=\operatorname{sech} i\theta,$$

and $\operatorname{cosec}\theta = i\operatorname{cosech} i\theta.$

From the above definitions, the following relations between the hyperbolic functions can be easily seen to be true :

$$e^{\theta}=\cosh\theta+\sinh\theta,\ e^{-\theta}=\cosh\theta-\sinh\theta,$$

$$\cosh^2\theta-\sinh^2\theta=1,$$

$$\operatorname{sech}^2\theta+\tanh^2\theta=1,$$

and $$\coth^2\theta-\operatorname{cosech}^2\theta=1.$$

The last three correspond to the relations $\cos^2\theta+\sin^2\theta=1$, $\sec^2\theta-\tan^2\theta=1$ and $\operatorname{cosec}^2\theta-\cot^2\theta=1$, between the ordinary circular functions.

6·11. Addition formulae. Addition formulae for the hyperbolic functions can also be deduced from the definitions of these functions. The formulae are

$$\cosh(u\pm v)=\cosh u\cosh v\pm\sinh u\sinh v,$$

$$\sinh(u\pm v)=\sinh u\cosh v\pm\cosh u\sinh v,$$

$$\tanh(u\pm v)=\frac{\tanh u\pm\tanh v}{1\pm\tanh u\tanh v},$$

and $$\coth(u\pm v)=\frac{\coth u\coth v\pm 1}{\coth v\pm\coth u}.$$

Alternative proof of any of these would be to use the corresponding formula for the ordinary trigonometrical functions. Thus, for the first,

$$\cosh(u+v) = \cos i(u+v)$$
$$= \cos iu \cos iv - \sin iu \sin iv$$
$$= \cosh u \cosh v + \sinh u \sinh v.$$

6·2. Periodicity of the hyperbolic functions. Since $e^{\theta} = e^{\theta+2n\pi i}$,
we have
$\cosh\theta = \cosh(\theta+2n\pi i)$ and $\sinh\theta = \sinh(\theta+2n\pi i)$, where n is any integer. Thus the period of hyperbolic sine and cosine is $2\pi i$.

Also, because of the equalities

$$e^{\theta+\pi i} = -e^{\theta} \text{ and } e^{-(\theta+\pi i)} = -e^{-\theta},$$

we have
$\cosh(\theta+\pi i) = -\cosh\theta$ and $\sinh(\theta+\pi i) = -\sinh\theta$,
giving that

$$\tanh(\theta+\pi i) = \tanh\theta$$
and
$$\coth(\theta+\pi i) = \coth\theta.$$

The period of hyperbolic tangent and cotangent is therefore πi, *i.e.*, half of the period of hyperbolic sine and cosine.

We give below the values of the hyperbolic functions for the arguments 0, $\frac{1}{2}\pi i$, πi, $\frac{3}{2}\pi i$.

	$\theta=0$	$\frac{1}{2}\pi i$	πi	$\frac{3}{2}\pi i$
$\sinh\theta$	0	i	0	$-i$
$\cosh\theta$	1	0	-1	0
$\tanh\theta$	0	$i\times\infty$	0	$i\times\infty$
$\coth\theta$	∞	0	∞	0
$\operatorname{sech}\theta$	1	∞	-1	∞
$\operatorname{cosech}\theta$	∞	$-i$	∞	i

6·3. The inverse hyperbolic functions. If $\sinh w = z$, then w is called the inverse hyperbolic sine of z, and is written as

$$w = \sinh^{-1} z.$$

The other inverse hyperbolic functions, viz. $\cosh^{-1} z$, $\tanh^{-1} z$, $\coth^{-1} z$, $\operatorname{sech}^{-1} z$ and $\operatorname{cosech}^{-1} z$ are defined similarly.

Since $\sinh w = -i \sin iw = z$, say,

we have $$\sinh^{-1} z = -i \sin^{-1}(iz).$$

Similarly, $$\cosh^{-1} z = -i \cos^{-1}(iz),$$

and $$\tanh^{-1} z = -i \tan^{-1}(iz).$$

These inverse hyperbolic functions can be given precise meanings by means of the logarithmic functions. Thus if

$$\sinh w = z,$$

we have $$e^{\omega} - e^{-\omega} = 2z,$$

or $$e^{2\omega} - 2ze^{\omega} - 1 = 0.$$

This is a quadratic equation in e^{ω} and so

$$e^{\omega} = z \pm \sqrt{(1+z^2)}.$$

Hence

$$w = 2n\pi i + \log_e \{z + \sqrt{(1+z^2)}\}$$
$$\text{or } 2n\pi i + \log_e \{z - \sqrt{(1+z^2)}\}.$$

Both these values of w are included in the expression $k\pi i + (-)^k \log\{z + \sqrt{(1+z^2)}\}$.

This gives that the general value of $\sinh^{-1} z$ is $k\pi i + (-)^k \log_e \{z + \sqrt{(1+z^2)}\}$ and its principal value is $\log_e \{z + \sqrt{(1+z^2)}\}$ and is denoted by $\sinh^{-1}(z)$.

Similarly, starting with $\cosh w = z$, i.e., $e^{\omega} + e^{-\omega} = 2z$, we get

$$e^{\omega} = z \pm \sqrt{(z^2-1)},$$

or $$w=2n\pi i\pm\log\{z+\sqrt{(z^2-1)}\}.$$

This is the general value of $\cosh^{-1}(z)$ and its principal value written as $\cosh^{-1}(z)$ is

$$\log\{z+\sqrt{(z^2-1)}\}.$$

Now let us suppose that $\tanh w=z$. Then

$$\frac{e^w-e^{-w}}{e^w+e^{-w}}=z, \text{ or } \frac{e^{2w}-1}{e^{2w}+1}=z.$$

It follows that

$$e^{2w}=\frac{1+z}{1-z},$$

or $$w=n\pi i+\tfrac{1}{2}\log_e\frac{1+z}{1-z}.$$

This is the general value of $\tanh^{-1}(z)$, the principal value being $\frac{1}{2}\log\dfrac{1+z}{1-z}$.

Similarly, the principal values of $\coth^{-1}(z)$, $\text{sech}^{-1}(z)$ and $\text{cosech}^{-1}(z)$ are respectively the expressions

$$\tfrac{1}{2}\log\frac{z+1}{z-1},\ \log_e\frac{1+\sqrt{(1-z^2)}}{z} \text{ and } \log_e\frac{1+\sqrt{(1+z^2)}}{z}.$$

6·4. The real and imaginary parts of circular functions. By the help of hyperbolic functions it is possible to express the circular functions with a complex argument, in the form $\alpha+i\beta$, where α and β are real. Thus

$$\sin(x\pm iy)=\sin x\cos iy\pm\cos x\sin iy.$$
$$=\sin x\cosh y\pm i\cos x\sinh y.$$

Similarly, $\cos(x\pm iy)=\cos x\cosh y\mp i\sin x\sinh y.$

Also $\quad \tan(x+iy) = \dfrac{\sin(x+iy)\cos(x-iy)}{\cos(x+iy)\cos(x-iy)}$

$$= \frac{\sin 2x + \sin 2iy}{\cos 2x + \cos 2iy}$$

$$= \frac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}.$$

Similarly, inverse trigonometrical functions can be expressed in the form $\alpha + i\beta$. Thus if

$$\cos^{-1}(x+iy) = \alpha - i\beta,$$

then $\quad \cos^{-1}(x-iy) = \alpha - i\beta,$

because if two complex numbers are equal, their conjugates must also be equal.

Hence $\cos 2\alpha = \cos\{(\alpha+i\beta)+(\alpha-i\beta)\}$

$$= \cos(\alpha+i\beta)\cos(\alpha-i\beta) - \sin(\alpha+i\beta)\sin(\alpha-i\beta)$$

$$= (x+iy)(x-iy) - \sqrt{\{1-(x+iy)^2\}}\sqrt{\{1-(x-iy)^2\}}$$

$$= x^2+y^2 - \sqrt{\{(1-x^2+y^2)-2ixy\}} \times \sqrt{\{(1-x^2+y^2)+2ixy\}},$$

$$= x^2+y^2 - \sqrt{\{(1-x^2+y^2)^2+4x^2y^2\}}.,$$

giving $\alpha = \frac{1}{2}\cos^{-1}[x^2+y^2-\sqrt{\{(1-x^2+y^2)^2+4x^2y^2\}}]$.

Also $\quad \cos 2i\beta = \cos\{(\alpha+i\beta)-(\alpha-i\beta)\}$

$$= x^2+y^2+\sqrt{\{(1-x^2+y^2)^2+4x^2y^2\}},$$

on proceeding as before. Since $\cos 2i\beta = \cosh 2\beta$, this gives that

$$\beta = \tfrac{1}{2}\cosh^{-1}[(x^2+y^2)+\sqrt{\{(1-x^2+y^2)^2+4x^2y^2\}}].$$

Ex. 1. If $\log\sin(\theta+i\phi) = \alpha+i\beta$, prove that

$2\cos 2\theta = e^{2\phi}+e^{-2\phi}-4e^{2\alpha}$ and $\cos(\theta-\beta) = e^{2\phi}\cos(\theta+\beta)$.

[*Jammu & Kashmir*, 1953]

Since $$\log \sin (\theta+i\phi)=\alpha+i\beta,$$

$\therefore$ $$\sin (\theta+i\phi)=e^{\alpha+i\beta},$$

or $$\sin \theta \cosh \phi+i \cos \theta \sinh \phi=e^{\alpha}(\cos \beta+i \sin \beta).$$

Equating the real and imaginary parts, we have

$$\sin \theta \cosh \phi=e^{\alpha} \cos \beta, \tag{1}$$

and $$\cos \theta \sinh \phi=e^{\alpha} \sin \beta. \tag{2}$$

Squaring and adding (1) and (2), we get that

$$\sin^2 \theta \cosh^2 \phi+\cos^2 \theta \sinh^2 \phi=e^{2\alpha}.$$

But $\sin^2 \theta=\frac{1}{2}(1-\cos 2\theta)$, $\cos^2 \theta=\frac{1}{2}(1+\cos 2\theta)$,

hence $$\tfrac{1}{2}(1-\cos 2\theta) \cosh^2 \phi+\tfrac{1}{2}(1+\cos 2\theta) \sinh^2 \phi=e^{2\alpha},$$

or $$\cosh^2 \phi+\sinh^2 \phi+\cos 2\theta\,(\sinh^2 \phi-\cosh^2 \phi)=2e^{2\alpha},$$

or $$\{\tfrac{1}{2}(e^{\phi}+e^{-\phi})\}^2+\{\tfrac{1}{2}(e^{\phi}-e^{-\phi})\}^2 +\cos 2\theta[\{\tfrac{1}{2}(e^{\phi}-e^{-\phi})\}^2-\{\tfrac{1}{2}(e^{\phi}+e^{-\phi})\}^2]=2e^{2\alpha}.$$

This gives, on simplification,

$$2 \cos 2\theta=e^{2\phi}+e^{-2\phi}-4e^{2\alpha}.$$

Again, dividing (2) by the corresponding sides of (1), we have

$$\frac{\cos \theta \sinh \phi}{\sin \theta \cosh \phi}=\frac{\sin \beta}{\cos \beta},$$

or $$\frac{\sinh \phi}{\cosh \phi}=\frac{\sin \theta \sin \beta}{\cos \theta \cos \beta},$$

i.e., $$\frac{\cosh \phi+\sinh \phi}{\cosh \phi-\sinh \phi}=\frac{\cos \theta \cos \beta+\sin \theta \sin \beta}{\cos \theta \cos \beta-\sin \theta \sin \beta},$$

or $$e^{2\phi}=\frac{\cos (\theta-\beta)}{\cos (\theta+\beta)},$$

giving that $$\cos (\theta-\beta)=e^{2\phi} \cos (\theta+\beta).$$

Ex. 2. Separate into real and imaginary parts the expression

$$\sin^{-1} (\cos \theta+i \sin \theta),$$

where θ is a positive acute angle. [*Lucknow*, 1960]

First Method. Let $\sin^{-1} (\cos \theta+i \sin \theta)=x+iy$. Then

$$\sin (x+iy)=\cos \theta+i \sin \theta, \tag{1}$$

and also $$\sin (x-iy)=\cos \theta-i \sin \theta, \tag{2}$$

because when two complex numbers are equal, their conjugates must also be equal.

Therefore $\cos 2x$

$$
\begin{aligned}
&=\cos\{(x+iy)+(x-iy)\}\\
&=\cos(x+iy)\cos(x-iy)-\sin(x+iy)\sin(x-iy)\\
&=\sqrt{1-(\cos\theta+i\sin\theta)^2}\sqrt{1-(\cos\theta-i\sin\theta)^2}\\
&\quad-(\cos\theta+i\sin\theta)(\cos\theta-i\sin\theta), \text{ using (1) and (2).}\\
&=\sqrt{1-\cos^2\theta+\sin^2\theta-2i\sin\theta\cos\theta}\\
&\qquad\times\sqrt{1-\cos^2\theta+\sin^2\theta+2i\sin\theta\cos\theta}-1\\
&=\sqrt{2\sin^2\theta-2i\sin\theta\cos\theta}\sqrt{2\sin^2\theta+2i\sin\theta\cos\theta}-1\\
&=\sqrt{2\sin\theta(\sin\theta-i\cos\theta)}\sqrt{2\sin\theta(\sin\theta+i\cos\theta)}-1\\
&=2\sin\theta-1.
\end{aligned}
$$

or $$2\cos^2 x-1=2\sin\theta-1,$$

or $$\cos^2 x=\sin\theta$$

or $$\cos x=\sqrt{(\sin\theta)},$$

the positive square root is taken because x, being the real part of an inverse sine, lies between $-\pi/2$ and $+\pi/2$(§5·7)

$\therefore$ $$x=\cos\sqrt{(\sin\theta)}.$$

Again $$\cos(2iy)=\cos\{(x+iy)-(x-iy)\}$$
$$=2\sin\theta+1, \text{ as before}$$

or $$2\cos^2(iy)-1=2\sin\theta+1.$$

$\therefore$ $$\cos(iy)=\sqrt{(1+\sin\theta)},$$

or $$\cosh y=\sqrt{1+\sin\theta}.$$

Therefore $$y=\cosh^{-1}\sqrt{1+\sin\theta}$$
$$=\log(\sqrt{1+\sin\theta}+\sqrt{\sin\theta}).$$

Hence $\sin^{-1}(\cos\theta+i\sin\theta)$

$$
\begin{aligned}
&=x+iy\\
&=\cos^{-1}\sqrt{(\sin\theta)}+i\log\{\sqrt{(\sin\theta)}+\sqrt{(1+\sin\theta)}\}.
\end{aligned}
$$

In general,

$$\sin^{-1}(\cos\theta+i\sin\theta)=n\pi+(-1)^n[\cos^{-1}\sqrt{(\sin\theta)}$$
$$+i\log\{\sqrt{(\sin\theta)}+\sqrt{(1+\sin\theta)}\}]$$

Second Method. Let $\sin^{-1}(\cos\theta+i\sin\theta)=x+iy$. Then $\cos\theta+i\sin\theta=\sin(x+iy)=\sin x\cosh y+i\cos x\sinh y$. Therefore

$$\sin x\cosh y=\cos\theta, \tag{1}$$

and

$$\cos x\sinh y=\sin\theta. \tag{2}$$

Squaring and adding (1) and (2), we have

$$\begin{aligned}1&=\sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y\\&=\sin^2 x(1+\sinh^2 y)+\cos^2 x\sinh^2 y\\&=\sin^2 x+\sinh^2 y.\end{aligned}$$

Therefore

$$\sinh^2 y=\cos^2 x.$$

Hence from (2) we have

$$\cos^2 x=\sin\theta$$

because, θ being a positive acute angle, $\sin\theta$ is positive.

Now as x is the real part of $\sin^{-1}(\cos\theta+i\sin\theta)$ it lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ (§ 5·7), and so we have

$$\cos x=\sqrt{(\sin\theta)},$$

i.e.,

$$x=\cos^{-1}\sqrt{(\sin\theta)}.$$

The relation (2) then gives

$$\sinh y=+\sqrt{(\sin\theta)};$$

so that

$$e^{2y}-2e^y\sqrt{(\sin\theta)}=1,$$

giving

$$e^y=\sqrt{(\sin\theta)}+\sqrt{(1+\sin\theta)},$$

i.e.,

$$y=\log\{\sqrt{(\sin\theta)}+\sqrt{(1+\sin\theta)}\}.$$

Hence, as before,

$$\sin^{-1}(\cos\theta+i\sin\theta)=\cos^{-1}\sqrt{(\sin\theta)}+i\log\{\sqrt{(\sin\theta)}+\sqrt{(1+\sin\theta)}\}.$$

Ex. 3. Prove that

$$\tan^{-1}(x+iy)=\tfrac{1}{2}\tan^{-1}\frac{2x}{1-x^2-y^2}+\tfrac{1}{2}i\tanh^{-1}\frac{2y}{1+x^2+y^2}.$$

[*Lucknow*, '65]

Let $\tan^{-1}(x+iy)=A+iB$. Then

$$\tan(A+iB)=x+iy, \tag{1}$$

and also
$$\tan(A-iB)=x-iy. \tag{2}$$

Therefore $\tan 2A=\tan\{(A+iB)+(A-iB)\}$

$$=\frac{\tan(A+iB)+\tan(A-iB)}{1-\tan(A+iB)\tan(A-iB)}$$

$$=\frac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)}, \text{ using (1) and (2)}$$

$$=\frac{2x}{1-x^2-y^2},$$

giving
$$A=\tfrac{1}{2}\tan^{-1}\frac{2x}{1-x^2-y^2}.$$

Also $\tan 2iB=\tan\{(A+iB)-(A-iB)\}$

$$=\frac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)}, \text{ using (1) and (2)}$$

$$=\frac{2iy}{1+x^2+y^2},$$

or
$$\tanh 2B=\frac{2y}{1+x^2+y^2},$$

giving
$$B=\tfrac{1}{2}\tanh^{-1}\frac{2y}{1+x^2+y^2}.$$

Hence $\tan^{-1}(x+iy)=\tfrac{1}{2}\tan^{-1}\dfrac{2x}{1-x^2-y^2}+\tfrac{1}{2}i\tanh^{-1}\dfrac{2y}{1+x^2+y^2}.$

Ex. 4. Prove that

$$\tan^{-1}(e^{i\theta})=\tfrac{1}{2}n\pi+\tfrac{1}{4}\pi+\tfrac{1}{2}i\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right).$$

[*Travancore*, '51; *Lucknow*, '56]

First Method. Let $\tan^{-1}(e^{i\theta})=A+iB$. Then

$$\tan(A+iB)=e^{\theta i}=\cos\theta+i\sin\theta, \tag{1}$$

and also

$$\tan(A-iB)=\cos\theta-i\sin\theta. \tag{2}$$

Therefore $$\tan 2A = \frac{\tan(A+iB)+\tan(A-iB)}{1-\tan(A+iB)\tan(A-iB)}$$

$$= \frac{(\cos\theta+i\sin\theta)+(\cos\theta-i\sin\theta)}{1-(\cos\theta+i\sin\theta)(\cos\theta-i\sin\theta)},$$

using (1) and (2)

$$= \frac{2\cos\theta}{1-1}$$

$$= \tan(n\pi+\tfrac{1}{2}\pi).$$

$$\therefore \quad 2A = n\pi+\tfrac{1}{2}\pi.$$

$$\therefore \quad A = \tfrac{1}{2}n\pi+\tfrac{1}{4}\pi.$$

Also $$\tan(2iB) = \tan\{(A+iB)-(A-iB)\}$$

$$= \frac{\tan(A+iB)-\tan(A-iB)}{1+\tan(A+iB)\tan(A-iB)}$$

$$= \frac{(\cos\theta+i\sin\theta)-(\cos\theta-i\sin\theta)}{1+(\cos\theta+i\sin\theta)(\cos\theta-i\sin\theta)},$$ using (1) and (2)

$$= \frac{2i\sin\theta}{2} \text{ or } i\sin\theta,$$

or $$\tanh 2B = \sin\theta,$$

or $$\frac{e^{2B}-e^{-2B}}{e^{2B}+e^{-2B}} = \frac{\sin\theta}{1}.$$

By Componendo and Dividendo

$$\frac{2e^{2B}}{2e^{-2B}} = \frac{1+\sin\theta}{1-\sin\theta} = \frac{1-\cos(\pi/2+\theta)}{1+\cos(\pi/2+\theta)}$$

or $$e^{4B} = \frac{2\sin^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right)} = \tan^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

$$\therefore \quad e^{2B} = \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right).$$

Therefore $B = \frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right).$

Hence

$$\tan^{-1}(e^{\theta i})=A+iB=\tfrac{1}{2}n\pi+\tfrac{1}{4}\pi+\tfrac{1}{2}i\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right).$$

Second Method. Let $\tan^{-1}(e^{i\theta})=z$. Then

$$\tan z=e^{i\theta}=\cos\theta+i\sin\theta,$$

or

$$\frac{e^{zi}-e^{-zi}}{i(e^{zi}+e^{-zi})}=\cos\theta+i\sin\theta,$$

or

$$\frac{e^{zi}-e^{-zi}}{e^{zi}+e^{-zi}}=i\cos\theta-\sin\theta.$$

Therefore, by Componendo and Dividendo,

$$\frac{2e^{zi}}{2e^{-zi}}=\frac{(1-\sin\theta)+i\cos\theta}{1+\sin\theta-i\cos\theta}$$

or

$$e^{2zi}=\frac{1-\sin\theta+i\cos\theta}{1+\cos\theta-i\cos\theta}.$$

Therefore

$$2zi=\log\{(1-\sin\theta)+i\cos\theta\}-\log\{(1+\sin\theta)-i\cos\theta\}$$

$$=\log\sqrt{(2-2\sin\theta)}+i\tan^{-1}\frac{\cos\theta}{1-\sin\theta}$$

$$-\log\sqrt{(2+2\sin\theta)}+i\tan^{-1}\frac{\cos\theta}{1+\sin\theta}$$

$$=\log\sqrt{\left\{\frac{1-\sin\theta}{1+\sin\theta}\right\}}$$

$$+i\tan^{-1}\left\{\left(\frac{\cos\theta}{1-\sin\theta}+\frac{\cos\theta}{1+\sin\theta}\right)\Big/\left(1-\frac{\cos^2\theta}{1-\sin^2\theta}\right)\right\}$$

$$=\log\sqrt{\left\{\frac{1+\cos\left(\frac{\pi}{2}+\theta\right)}{1-\cos\left(\frac{\pi}{2}+\theta\right)}\right\}}$$

$$+i\tan^{-1}\left\{\tan\left(n\pi+\frac{\pi}{2}\right)\right\}$$

$$=\log\sqrt{\left\{\frac{2\cos^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}{2\sin^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}\right\}}+i\left(n\pi+\frac{\pi}{2}\right)$$

$$=\log\cot\left(\frac{\pi}{4}+_2\right)+i\left(n\pi+\frac{\pi}{2}\right)$$

$$=i\left(n\pi+\frac{\pi}{2}\right)-\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right).$$

Hence $\quad z\equiv\tan^{-1}(e^{\theta i})=\frac{1}{2}n\pi+\frac{1}{4}\pi+\frac{1}{2}i\log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right).$

EXAMPLES

1. Prove the following statements :

(*i*) $\cosh 2\theta=2\cosh^2\theta-1=1+2\sinh^2\theta$. [*Poona*, 1958]

(ii) $\sinh 2\theta=\dfrac{2\tanh\theta}{1-\tanh^2\theta}$ and $\cosh 2\theta=\dfrac{1+\tanh^2\theta}{1-\tanh^2\theta}$,

(iii) $\tanh 2\theta=\dfrac{2\tanh\theta}{1+\tanh^2\theta}$,

(iv) $\sinh 3\theta=4\sinh^3\theta+3\sinh\theta$,

and $\quad\cosh 3\theta=4\cosh^3\theta-3\cosh\theta$.

(v) $\tanh 3\theta=\dfrac{3\tanh\theta+\tanh^3\theta}{1+3\tanh^2\theta}$.

(vi) $\dfrac{1+\tanh z}{1-\tanh z}=\cosh 2z+\sinh 2z$. [*Travan.*, 1947]

(vii) $\left(\dfrac{1+\tanh\theta}{1-\tanh\theta}\right)^6=\cosh 12\theta+\sinh 12\theta$. [*Andhra*, '37]

(viii) $(\cosh\theta\pm\sinh\theta)^n=\cosh n\theta\pm\sinh n\theta$.

(ix) $\cosh n\theta=\cosh^n\theta+\dfrac{n(n-1)}{2\,!}\cosh^{n-2}\theta\sinh^2\theta+\ldots$.

(x) $\sinh n\theta=n\cosh^{n-1}\sinh\theta+\dfrac{n(n-1)(n-2)}{3\,!}\cosh^{n-3}\theta$
$\times\sinh^3\theta+\ldots$.

(xi) $\sinh^{-1}x=\cosh^{-1}\sqrt{(1+x^2)}=\tanh^{-1}\dfrac{x}{\sqrt{(1+x^2)}}$.

(xii) $\dfrac{\sinh(n+1)u-\sinh(n-1)u}{\cosh(n+1)u+2\cosh nu+\cosh(n-1)u}=\tanh\dfrac{u}{2}$.
[*Andhra*, 1932]

2. If $\tanh x = 0{\cdot}75$, find x. [*Madras*, 1931]

3. Find $\lim_{\theta \to 0} \dfrac{\sinh \theta - \sin \theta}{\theta(\cosh \theta - \cos \theta)}$. [*Travancore*, 1944]

4. Show that

(i) $\sinh^{-1}(\cot x) = \log(\cot x + \operatorname{cosec} x)$,

(ii) $\log \tan\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right) = 2 \tanh^{-1}\left(\tan \dfrac{\theta}{2}\right)$,

(iii) $\log \sec x = \coth^{-1}(2 \operatorname{cosec}^2 x - 1)$, if $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$.

5. Separate the following into real and imaginary parts :

(i) $\cosh(x+iy)$, [*Delhi*, 1951]

(ii) $\cot(x+iy)$, [*Alld.*, 1951; *Baroda*, 1952]

(iii) $\tanh(x+iy)$, [*Agra*, 1945]

(iv) $\sec(x+iy)$,

(v) $\sin^2(x+iy)$,

(vi) $e^{\sin(\alpha+i\beta)}$. [*Mysore*, 1945]

6. Separate $\log \log i$ and $\log \cos(x+iy)$ each into real and imaginary parts. [*Bombay*, 1934]

7. Break up the following functions into real and imaginary parts—

(i) $\tan^{-1}(x+iy) - \tan^{-1}(x-iy)$,

(ii) $\cosh \log(x+iy) + \cosh \log(x-iy)$. [*Andhra*, 1937]

8. Show that the sum of the moduli of the values of $(1+i)^{1+i}$, which are less than unity, is

$$(1/\sqrt{2})e^{3\pi/4} \operatorname{cosech} \pi. \quad [\textit{Lucknow}, 1956]$$

9. If $\operatorname{cosec} e^{\theta i} = x - iy$, show that

$$\frac{y}{x} = \tan(\cos\theta) \cdot \frac{e^{\sin\theta} + e^{-\sin\theta}}{e^{\sin\theta} - e^{-\sin\theta}}.$$

10. Prove that

$$\log \tan\left(\frac{\pi}{4} + \frac{ix}{2}\right) = i \tan^{-1} \sinh x.$$

[*Banaras*, 1954; *Luck.*, 1968]

11. If u and v are defined by the equation

$$u+iv=\tan(x+iy),$$

where x, y, u, v are all real, express u and v in terms of x and y and prove that

$$u^2+v^2=\frac{\cosh^2 y-\cos^2 x}{\cosh^2 y-\sin^2 x}.$$

12. Prove that

$$\log\cos(\xi+i\eta)=A+iB,$$

where $$A=\tfrac{1}{2}\log\{\tfrac{1}{2}(\cosh 2\eta+\cos 2\xi)\},$$

and B is any angle such that

$$\frac{\cos B}{\cos\xi\cosh\eta}=\frac{-\sin B}{\sin\xi\sinh\eta}=\frac{1}{\sqrt{\{\frac{1}{2}(\cosh 2\eta+\cos 2\xi)\}}}.$$

13. If $\sin(A+iB)=x+iy$, prove that

$$\frac{x^2}{\cosh^2 B}+\frac{y^2}{\sinh^2 B}=1 \quad\text{and}\quad \frac{x^2}{\sin^2 A}-\frac{y^2}{\cos^2 A}=1.$$

[*Agra*, 1959; *Luck.*, 1967]

14. If $\tan y=\tan\alpha\tanh\beta$ and $\tan z=\cot\alpha\tanh\beta$, prove that

$$\tan(y+z)=\sinh 2\beta\operatorname{cosec} 2\alpha.$$ [*Panjab*, 1944]

15. If $\tan(\theta+i\phi)=\sin(x+iy)$, prove that

$$\coth y\sinh 2\phi=\cot x\sin 2\theta.$$

[*Lucknow*, 1964]

16. If $\cos(\alpha+i\beta)=\cos\phi+i\sin\phi$, show that

$$\sin\phi=\pm\sin^2\alpha=\pm\sinh^2\beta.$$ [*Poona*, 1957]

17. If $\tan(A+iB)=\tan\theta+i\sec\theta$, show that

$$e^{2B}=\pm\cot\tfrac{1}{2}\theta,$$

and that $$2A=n\pi+\tfrac{1}{2}\pi+\theta.$$ [*Lucknow*, 1955]

18. If $\tan(\alpha+i\beta)=x+iy$, prove that

(i) $x^2+y^2+2x\cot 2\alpha=1,$ [*Agra*, 1953]

(ii) $x^2+y^2-2y\coth 2\beta+1=0.$ [*Lucknow*, 1964]

19. If $\cosh(u+iv)=x+iy$, show that

$$\frac{x^2}{\cosh^2 u}+\frac{y^2}{\sinh^2 u}=1, \quad \text{and} \quad \frac{x^2}{\cos^2 v}-\frac{y^2}{\sin^2 v}=1.$$

[*Mysore*, 1936]

20. If $\sin(\theta+i\phi)=\tan\alpha+i\sec\alpha$, show that $\cos 2\theta\cosh 2\phi=3$. [*Lucknow*, '66]

21. If $\cos(x+iy)=\cos\theta+i\sin\theta$, show that $\cos 2x+\cosh 2y=2$. [*Lucknow*, '63]

22. If $\cos^{-1}(\alpha+i\beta)=\theta+i\phi$, show that
(i) $\alpha^2\operatorname{sech}^2\phi+\beta^2\operatorname{cosech}^2\phi=1$,
and (ii) $\alpha^2\sec^2\theta-\beta^2\operatorname{cosec}^2\theta=1$. [*Agra*, 1957]

23. If $\cos(\theta+i\phi)\,.\,\cos(\alpha+i\beta)=1$, prove that

$$\tanh^2\phi\cosh^2\beta=\sin^2\alpha,$$
$$\tanh^2\beta\cosh^2\phi=\sin^2\theta.$$

[*Bombay*, 1931]

[Hint. Writing the given relation as

$$\cos(\theta+i\phi)=1/\cos(\alpha+i\beta), \text{ we get}$$

$$\cos\theta\cosh\phi-i\sin\theta\sinh\phi=2(\cos\alpha\cosh\beta+i\sin\alpha\sinh\beta)/N, \quad (1)$$

where $N=\cos 2\alpha+\cosh 2\beta$. Also

$$\sin(\theta+i\phi)=\sqrt{\{1-1/\cos^2(\alpha+i\beta)\}}.$$

$$\therefore \quad \sin\theta\cosh\phi+i\cos\theta\sinh\phi=i(\sin 2\alpha+i\sin 2\beta)/N. \quad (2)$$

From (1) and (2), we get

$$\cos\theta\cosh\phi=(2\cos\alpha\cosh\beta)/N \quad (3)$$

and

$$\cos\theta\sinh\phi=(\sin 2\alpha)/N. \quad (4)$$

Dividing (4) by (3), then squaring and transposing,

$$\tanh^2\phi\cosh^2\beta=\sin^2\alpha.$$

To prove the second result, write the given relation as

$$\cos(\alpha+i\beta)=1/\cos(\theta+i\phi),$$

and proceed as above.]

24. If $[\cos(\alpha+i\beta)]^{p+iq}=A+iB$, show that

$$\tan^{-1}\frac{B}{A}=\frac{q}{2}\log(\cosh^2\beta-\sin^2\alpha)-p\tan^{-1}(\tan\alpha\tanh\beta).$$

[*Mysore*, 1932]

25. If $\cos^{-1}(u+iv)=\alpha+i\beta$, prove that $\cos^2\alpha$ and $\cosh^2\beta$ are the roots of the equation

$$x^2-x(1+u^2+v^2)+u^2=0. \qquad [\textit{Calcutta}, 1938]$$

26. If $X+iY=\log\dfrac{z+1}{z-1}$, where $z=x+iy$, show that $X=$const. represents a system of coaxial circles which cut orthogonally the coaxial system given by $Y=$const.

[*I. C. S.*, 1932]

27. Given that x, y, u, v are all real and that

$$x+iy=\cos(u+iv),$$

prove that

$$(1+x)^2+y^2=(\cosh v+\cos u)^2,$$

$$(1-x)^2+y^2=(\cosh v-\cos u)^2.$$

Prove further that if $x=\cos\theta$, $y=\sin\theta$, where $0<\theta<\pi$, then

$$\cosh v=\cos\tfrac{1}{2}\theta+\sin\tfrac{1}{2}\theta,$$

$$\cos u=\cos\tfrac{1}{2}\theta-\sin\tfrac{1}{2}\theta.$$

28. Show that if

$$\cosh^{-1}(x+iy)+\cosh^{-1}(x-iy)=\cosh^{-1}a,$$

then

$$2(a-1)x^2+2(a+1)y^2=a^2-1. \qquad [\textit{Bombay}, 1931]$$

29. Separate into its real and imaginary parts

$$\cos^{-1}(\cos\theta+i\sin\theta),$$

where θ is a positive acute angle. [*Panjab*, '44; *Poona*, '56]

80. Prove that one of the values of

$$\sin^{-1}\left(\frac{5\sqrt{7}+9i}{16}\right)$$

is $\cos^{-1}(3/4)+i\log 2$. [*Andhra*, 1937]

CHAPTER VII

EXPANSIONS OF TRIGONOMETRICAL FUNCTIONS

7·1. Introduction. In this chapter we shall give the important expansions of some of the trigonometrical functions. To begin with, we deduce those which depend on the results of the next section.

7·2. Cos $n\theta$ and sin $n\theta$ in terms of z. Let

$$z = \cos\theta + i\sin\theta.$$

Then $\quad z^{-1} = (\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta.$

On adding and subtracting, we get that

$$z + z^{-1} = 2\cos\theta \text{ and } z - z^{-1} = 2i\sin\theta. \qquad (1)$$

Further, by De Moivre's theorem,

$$z^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$$

and $\quad z^{-n} = (\cos\theta + i\sin\theta)^n = \cos n\theta - i\sin n\theta,$

where n is an integer. These give

$$z^n + z^{-n} = 2\cos n\theta \text{ and } z^n - z^{-n} = 2i\sin n\theta. \qquad (2)$$

7·21. Expansion of $\cos^n\theta$. *To express $\cos^n\theta$ in a series of cosines of multiples of θ, when n is a positive integer.*

From (1) of § 7·2, we have

$$\begin{aligned}(2\cos\theta)^n &= (z + z^{-1})^n \\ &= z^n + {}^nC_1 z^{n-2} + \ldots + {}^nC_{n-1} z^{-(n-2)} + z^{-n} \qquad (3) \\ &= (z^n + z^{-n}) + {}^nC_1(z^{n-2} + z^{-(n-2)}) \\ &\qquad + {}^nC_2(z^{n-4} + z^{-(n-4)}) + \ldots,\end{aligned}$$

on rearranging the series, taking the first and the last terms together, then the second and the last but one, and so on. Hence, on using (2) of § 7·2, we get

$$(2\cos\theta)^n = 2\cos n\theta + 2n\cos(n-2)\theta + 2\frac{n(n-1)}{2!}\cos(n-4)\theta + \ldots$$

or $$2^{n-1}\cos^n\theta = \cos n\theta + n\cos(n-2)\theta + \frac{n(n-1)}{2!}\cos(n-4)\theta + \ldots . \qquad (4)$$

The last term of this series will assume different forms according as n is even or odd. When n is even, there will be an odd number of terms in (3), and the last term of (4) will be the $(\frac{1}{2}n+1)$th term of (3). Hence, when n is even, the last term of (4) will be $\frac{1}{2}\,{}^nC_{n/2}$. When n is odd, the number of terms in (3) will be even, and the $\frac{1}{2}(n+1)$th and the $\frac{1}{2}(n+3)$th terms of (3) will form the last term of (4). Hence, when n is odd, the last term of (4) will be ${}^nC_{(n-1)/2}\cos\theta$.

Ex. 1. Expand $\cos^{10}\theta$ in a series of cosines of multiples of θ.

We have

$$\begin{aligned}(2\cos\theta)^{10} &= (z+z^{-1})^{10}\\ &= z^{10}+10z^8+45z^6+120z^4+210z^2+252\\ &\qquad +210z^{-2}+120z^{-4}+45z^{-6}+10z^{-8}+z^{-10}\\ &= (z^{10}+z^{-10})+10(z^8+z^{-8})+45(z^6+z^{-6})\\ &\qquad +120(z^4+z^{-4})+210(z^2+z^{-2})+252.\\ &= 2\cos 10\theta+20\cos 8\theta+90\cos 6\theta+240\cos 4\theta\\ &\qquad +420\cos 2\theta+252.\end{aligned}$$

Hence

$$\cos^{10}\theta = (\tfrac{1}{2})^9[\cos 10\theta+10\cos 8\theta+45\cos 6\theta+120\cos 4\theta+210\cos 2\theta+126].$$

Ex. 2. Express $\cos^9\theta$ in a series of cosines of multiples of θ. [*Annamalai*, 1947]

Here

$$\begin{aligned}(2\cos\theta)^9 &= (z+z^{-1})^9\\ &= z^9+9z^7+36z^5+84z^3+126z+126z^{-1}+84z^{-3}\\ &\qquad +36z^{-5}+9z^{-7}+z^{-9}\\ &= (z^9+z^{-9})+9(z^7+z^{-7})+36(z^5+z^{-5})\\ &\qquad +84(z^3+z^{-3})+126(z+z^{-1})\\ &= 2\cos 9\theta+18\cos 7\theta+72\cos 5\theta+168\cos 3\theta+252\cos\theta.\end{aligned}$$

Hence

$$\cos^9\theta = (\tfrac{1}{2})^8[\cos 9\theta+9\cos 7\theta+36\cos 5\theta +84\cos 3\theta+126\cos\theta].$$

7·22. Expansion of $\sin^n\theta$. *To express $\sin^n\theta$ in a series of sines or cosines of multiples of θ, when n is a positive integer.*

We have from (1) that

$$\begin{aligned}(2i\sin\theta)^n &= (z-z^{-1})^n\\ &= z^n+nz^{n-1}(-z^{-1})+\frac{n(n-1)}{1\,.\,2}z^{n-2}(-z^{-1})^2+\ldots\\ &\quad +\frac{n(n-1)}{1\,.\,2}z^2(-z^{-1})^{n-2}+nz(-z^{-1})^{n-1}\\ &\qquad +(-z^{-1})^n. \qquad (5)\end{aligned}$$

The sign of the last term in this series depends upon whether n is even or odd. Hence we take up the two cases separately.

CASE 1. Let n be even.

In this case

$(-z^{-1})^n = z^{-n}$, $(-z^{-1})^{n-1} = -z^{-n+1}$, and so on;

and $\qquad i^n = (-)^{n/2}$

The equation (5), therefore, can be written as

$$2^n(-)^{n/2}\sin^n\theta = z^n - nz^{n-2} + \frac{(n-1)}{1\,.\,2}z^{n-4} - \dots$$
$$+\frac{n(n-1)}{1\,.\,2}z^{-n+4} - nz^{-n+2} + z^{-n}$$
$$=(z^n+z^{-n}) - n(z^{n-2}+z^{-(n-2)})$$
$$+\frac{n(n-1)}{1\,.\,2}(z^{n-4}+z^{-(n-4)}) - \dots$$
$$=2\cos n\theta - 2n\cos(n-2)\theta$$
$$+2\frac{n(n-1)}{1\,.\,2}\cos(n-4)\theta - \dots.$$

Hence, when n is even,

$$2^{n-1}(-)^{n/2}\sin^n\theta = \cos n\theta - n\cos(n-1)\theta$$
$$+\frac{n(n-1)}{1\,.\,2}\cos(n-4)\theta - \dots\,. \tag{6}$$

It is clear that the last term here will be half of the $(\frac{1}{2}n+1)$th term of (5) and this is easily seen to be $\frac{1}{2}(-)^{n/2}\,.\,{}^nC_{n/2}$.

Case II. Let n be odd.

In this case

$$(-z^{-1})^n = -z^{-n},\quad (-z^{-1})^{n-1} = z^{-n+1},\ \text{and so on;}$$

and

$$i^n = i\,.\,i^{n-1} = i(-)^{(n-1)/2}.$$

The equation (5), therefore, can be written as

$$2^n\, i(-)^{(n-1)/2}\sin^n\theta = z^n - nz^{n-2} + \frac{n(n-1)}{1\,.\,2}z^{n-4} - \dots$$
$$-\frac{n(n-1)}{1\,.\,2}z^{-n+4} + nz^{-n+2} - z^{-n}$$

$$=(z^n-z^{-n})-n(z^{n-2}-z^{-n+2})$$
$$+\frac{n(n-1)}{1.2}(z^{n-4}-z^{-n+4})-\dots$$
$$=2i\sin n\theta-2ni\sin(n-2)\theta$$
$$+2i\frac{n(n-1)}{1.2}\sin(n-4)\theta-\dots.$$

Therefore, when n is odd,

$$2^{n-1}(-)^{(n-1)/2}\sin^n\theta=\sin n\theta-n\sin(n-2)\theta$$
$$+\frac{n(n-1)}{1.2}\sin(n-4)\theta-\dots.$$

The last term in this series will be half the sum of the $\frac{1}{2}(n+1)$th and $\frac{1}{2}(n+3)$th terms in (4), i.e.,

$$(-)^{(n-1)/2}\,{}^nC_{(n-1)/2}\sin\theta.$$

Ex. 1. Express $\sin^8\theta$ in a series of cosines of multiples of θ.

We have

$$(2i\sin\theta)^8=(z-z^{-1})^8$$
$$=z^8-8z^6+28z^4-56z^2+70-56z^{-2}+28z^{-4}-8z^{-6}+z^{-8}$$
$$=(z^8+z^{-8})-8(z^6+z^{-6})+28(z^4+z^{-4})-56(z^2+z^{-2})+70$$
$$=2\cos 8\theta-16\cos 6\theta+56\cos 4\theta-112\cos 2\theta+70.$$

Since $\qquad i^8=(i^2)^4=(-)^4=+1,$

we get

$$\sin^8\theta=\frac{1}{2^7}[\cos 8\theta-8\cos 6\theta+28\cos 4\theta-56\cos 2\theta+35].$$

Ex 2. Express $\sin^9\theta$ in a series of sines of multiples of θ.

Here

$$(2i\sin\theta)^9=(z-z^{-1})^9$$
$$=z^9-9z^7+36z^5-84z^3+126z-126z^{-1}$$
$$+84z^{-3}-36z^{-5}+9z^{-7}-z^{-9}$$

$$=(z^9-z^{-9})-9(z^7-z^{-7})+36(z^5-z^{-5})$$
$$-84(z^3-z^{-3})+126(z-z^{-1})$$
$$=2i\sin 9\theta-18i\sin 7\theta+72i\sin 5\theta.$$
$$-168i\sin 3\theta+252i\sin\theta.$$

But $i^9=i(i^2)^4=i.$

Hence

$$\sin^9\theta=(\tfrac{1}{2})^8[\sin 9\theta-9\sin 7\theta+36\sin 5\theta-84\sin 3\theta+126\sin\theta].$$

7·23. Expansion of $\sin^m\theta\cos^n\theta$. The method of the above sections can also be used to expand expressions of the form $\sin^m\theta\cos^n\theta$. We illustrate this by an example.

Ex. Expand $\sin^7\theta\cos^2\theta$ in a series of sines of multiples of θ.

We have

$$(2i\sin\theta)^7(2\cos\theta)^3=(z-z^{-1})^7(z+z^{-1})^3.$$

To obtain the coefficients of the various powers of z in the last product, we first write the coefficients in the expansion of $(z-z^{-1})^7$ and then multiply by $(z+z^{-1})$ three times in succession, as in the following scheme* :

1	−7	21	−35	35	−21	7	−1			
1	−6	14	−14	0	14	−14	6	−1		
1	−5	8	0	−14	14	0	−8	5	−1	
1	−4	3	8	−14	0	14	−8	−3	4	−1

giving

$$(z-z^{-1})^7(z+z^{-1})^3=z^{10}-4z^8+3z^6+8z^4-14z^2+14z^{-2}-8z^{-4}$$
$$-3z^{-6}+4z^{-8}-z^{-10}$$

*The numbers in the first row are the coefficients in the expansion of $(z-z^{-1})^7$ in descending powers of z; those in the second row are the coefficients obtained by multiplying $(z-z^{-1})^7$ by $(z+z^{-1})$; those in the third row are the coefficients obtained by multiplying $(z-z^{-1})^7$ by $(z+z^{-1})^2$; and those in the fourth row are the coefficients obtained by multiplying $(z-z^{-1})^7$ by $(z+z^{-})^3$.

$$=(z^{10}-z^{-10})-4(z^8-z^{-8})+3(z^6-z^{-6})+8(z^4-z^{-4})-14(z^2-z^{-2})$$

$$=2i\sin 10\theta-8i\sin 8\theta+6i\sin 6\theta+16i\sin 4\theta-28i\sin 2\theta.$$

Since $i^7=-i$, therefore we get that

$$\sin^7\cos^3\theta=-(\tfrac{1}{2})^9[\sin 10\theta-4\sin 8\theta+3\sin 6\theta+8\sin 4\theta-14\sin 2\theta].$$

NOTE. It may be noted that the expansion of $\sin^m\theta\cos^n\theta$ will be a series of cosines or sines of multiples of θ according as m is even or odd.

EXAMPLES

1. Prove that

(i) $\cos^3\theta=\frac{1}{4}[\cos 3\theta+3\cos\theta]$

and (ii) $\sin^3\theta=-\frac{1}{4}[\sin 3\theta-3\sin\theta]$.

2. Prove that

(i) $\sin^4\theta=(\frac{1}{2})^3[\cos 4\theta-4\cos 2\theta+3]$.

and (ii) $\cos^4\theta=(\frac{1}{2})^3[\cos 4\theta+4\cos 2\theta+3]$.

3. Prove that

$$16\sin^5\theta=\sin 5\theta-5\sin 3\theta+10\sin\theta.$$ [*Andhra*, '52]

4. Prove that

$$32\cos^6\theta=\cos 6\theta+6\cos 4\theta+15\cos 2\theta+10.$$ [*Madras*, '46]

5. Prove that

$$\cos^7\theta=(\tfrac{1}{2})^6[\cos 7\theta+7\cos 5\theta+21\cos 3\theta+35\cos\theta].$$ [*Delhi*, 1962]

and $$\sin^7\theta=-(\tfrac{1}{2})^6[\sin 7\theta-7\sin 5\theta+21\sin 3\theta-35\sin\theta].$$

6. Prove that

$$64(\cos^8\theta+\sin^8\theta)=\cos 8\theta+28\cos 4\theta+35.$$

7. Express $\cos^4\theta\sin^3\theta$ in a series of sines of multiples of θ. [*Travancore*, 1947]

8. Express $\sin^5\theta\cos^2\theta$ in a series of sines of multiples of θ. [*Banaras*, '62]

9. Express $\sin^6\theta\cos^2\theta$ in terms of cosines of multiples of θ. [*Lucknow*, '68]

10. Express $\cos^5\theta \sin^3\theta$ in terms of sines of multiples of θ. [*Travancore*, '51]

11. Expand $\cos^5\theta \sin^7\theta$ in a series of sines of multiples of θ. [*Agra*, 1941]

7·3. Two important expansions. We shall now deduce two important expansions, viz.

$$\frac{1-r\cos\theta}{1-2r\cos\theta+r^2}=1+r\cos\theta+r^2\cos 2\theta+\ldots$$
$$+r^n\cos n\theta+\ldots \qquad (1)$$

and
$$\frac{r\sin\theta}{1-2r\cos\theta+r^2}=r\sin\theta+r^2\sin 2\theta+\ldots$$
$$+r^n\sin n\theta+\ldots, \qquad (2)$$

which will be found to be very useful in getting the expansions of $\cos n\theta$ and $\sin n\theta$ in powers of $\cos\theta$ or $\sin\theta$ (§§ 7·41, 7·42, etc.)

We first note that the series on the right of (1) and (2) are both absolutely convergent when the positive value of r is less than unity. This follows from the Ratio-test of convergence, keeping in view the fact that

$$|\cos n\theta|\leqslant 1 \text{ and also } |\sin n\theta|\leqslant 1.$$

We shall give two proofs for the expansions.

First Proof. We have to prove the truth of (1). Now when we multiply each side of (1) by $1-2r\cos\theta+r^2$, we see that we have to prove that

$$1-r\cos\theta=(1-2r\cos\theta+r^2)(1+r\cos\theta+r^2\cos 2\theta$$
$$+\ldots+r^n\cos n\theta+\ldots). \qquad (3)$$

If we now perform the multiplication of the two expressions on the right, we see that

the term independent of r is 1,

the coefficient of r is $-2\cos\theta+\cos\theta=-\cos\theta$,
and the coefficient of r^n $(n>1)$ is

$$\cos n\theta-2\cos\theta\cos(n-1)\theta+\cos(n-2)\theta=0,$$

since $\cos n\theta+\cos(n-2)\theta=2\cos(n-1)\theta\cos\theta$.

This establishes (3) and consequently (1).

To prove the second result, we should show that

$$r\sin\theta=(1-2r\cos\theta+r^2)(r\sin\theta+r^2\sin 2\theta+\ldots).$$

On performing the multiplication on the right, we see that

the coefficient of r is $\sin\theta$,

and the coefficient of r^n $(n>1)$ is

$$\sin n\theta-2\cos\theta\sin(n-1)\theta+\sin(n-2)\theta,$$

which is zero. Hence the proposition.

SECOND PROOF. The infinite series

$$1+z+z^2+\ldots$$

converges if the modulus of z is less than unity and its sum is, on taking $z=r(\cos\theta+i\sin\theta)$,

$$\frac{1}{1-z}=\frac{1}{1-r(\cos\theta+i\sin\theta)}$$

$$=\frac{1-r\cos\theta+ir\sin\theta}{(1-r\cos\theta-ir\sin\theta)(1-r\cos\theta+ir\sin\theta)}$$

$$=\frac{1-r\cos\theta+ir\sin\theta}{1-2r\cos\theta+r^2}.$$

Therefore

$$1+z+z^2+\ldots+z^n+\ldots$$

$$=\frac{1-r\cos\theta+ir\sin\theta}{1-2r\cos\theta+r^2}. \qquad (i)$$

If we now use the fact that

$$z^n=r^n(\cos\theta+i\sin\theta)^n=r^n(\cos n\theta+i\sin n\theta),$$

and equate the real and imaginary parts in (i) we arrive at the expansions (1) and (2).

6·41. Expansion of cos $n\theta$ in a series of descending powers of cos θ.

From (1) of 7·3, we see that

$$\cos n\theta = \text{coefficient of } r^n \text{ in } (1-r\cos\theta) \times (1-2r\cos\theta+r^2)^{-1}$$

$$= \text{coefficient of } r^n \text{ in } (1-2r\cos\theta+r^2)^{-1}$$
$$- \text{coefficient of } r^{n-1} \text{ in } \cos\theta\,(1-2r\cos\theta+r^2)^{-1}.$$

But

$$(1-2r\cos\theta+r^2)^{-1} = \{1-r(2\cos\theta-r)\}^{-1}$$
$$= 1+r(2\cos\theta-r)+r^2(2\cos\theta-r)^2+\ldots\ldots + r^n(2\cos\theta-r)^n+\ldots\ldots$$

It is clear that the term

$$r^k\,(2\cos\theta-r)^k,\ k>n,$$

will involve terms containing r^{n+1}, r^{n+2}, etc., and the coefficient of r^n

in $r^n(2\cos\theta-r)^n$ is $(2\cos\theta)^n$,

in $r^{n-1}(2\cos\theta-r)^{n-1}$ is $-(n-1)(2\cos\theta)^{n-2}$,

in $r^{n-2}(2\cos\theta-r)^{n-2}$ is $\dfrac{(n-2)(n-3)}{1.2}(2\cos\theta)^{n-4}$,

and so on.

Hence the coefficient of r^n in $(1-2r\cos\theta+r^2)^{-1}$ is

$$(2\cos\theta)^n-(n-1)(2\cos\theta)^{n-2} + \frac{(n-2)(n-3)}{1.2}(2\cos\theta)^{n-4}-\ldots;$$

and accordingly the coefficient of

$$r^{n-1} \text{ in } \cos\theta\,(1-2r\cos\theta+r^2)^{-1}$$

is $$\cos\theta\{(2\cos\theta)^{n-1}-(n-2)(2\cos\theta)^{n-3}+\frac{(n-3)(n-4)}{1.2}(2\cos\theta)^{n-5}-\ldots\},$$

i.e., $$\tfrac{1}{2}\{(2\cos\theta)^{n}-(n-2)(2\cos\theta)^{n-2}+\frac{(n-3)(n-4)}{1.2}(2\cos\theta)^{n-4}+\ldots\}.$$

It follows that the coefficient of r^n in $(1-r\cos\theta)(1-2r\cos\theta+r^2)^{-1}$ is

$$[(2\cos\theta)^n-(n-1)(2\cos\theta)^{n-2}+\frac{(n-2)(n-3)}{1.2}\times(2\cos\theta)^{n-4}-\ldots]-\tfrac{1}{2}[(2\cos\theta)^n-(n-2)(2\cos\theta)^{n-2}+\frac{(n-3)n-4)}{1.2}(2\cos\theta)^{n-4}-\ldots]$$

$$=\tfrac{1}{2}[(2\cos\theta)^n-n(2\cos\theta)^{n-2}+\left\{2\,\frac{(n-2)(n-3)}{1.2}-\frac{(n-3)(n-4)}{1.2}\right\}(2\cos\theta)^{n-4}-\ldots]$$

$$=\tfrac{1}{2}[(2\cos\theta)^n-n(2\cos\theta)^{n-2}+\{n(n-3)/2!\}(2\cos\theta)^{n-4}-\ldots].$$

Hence $$2\cos n\theta=(2\cos\theta)^n-n(2\cos\theta)^{n-2}+\{n(n-3)/2!\}(2\cos\theta)^{n-4}-\ldots, \quad (1)$$

the general term being

$$(-1)^r\frac{n(n-r-1)(n-r-2)\ldots(n-2r+1)}{r!}(2\cos\theta)^{n-2r}.$$

The series on the right-hand side will continue so long as the powers of $2\cos\theta$ are not negative. The last term will be

$$(-)^{(n-)/2}\,.\,n(2\cos\theta) \text{ or } (-)^{n/2}\,.\,2,$$

according as n is odd or even.

7·42. Expansion of sin $n\theta$/sin θ in a series of descending powers of cos θ.

The equation (2) of § 7·3 can be written as

$$(1-2r\cos\theta+r^2)^{-1}=\sum_{n=1}^{\infty} r^{n-1}(\sin n\theta/\sin\theta).$$

This shows that $(\sin n\theta/\sin\theta)$ is equal to the coefficient of r^{n-1} in $(1-2r\cos\theta+r^2)^{-1}$.

But

$$\begin{aligned}(1-2r\cos\theta+r^2)^{-1}&=\{1-r(2\cos\theta-r)\}^{-1}\\ &=1+r(2\cos\theta-r)+r^2(2\cos\theta-r)^2+\ldots\\ &\quad+r^{n-1}(2\cos\theta-r)^{n-1}+\ldots.\end{aligned}$$

As in § 7·41, the coefficient of r^{n-1}

in $r^{n-1}(2\cos\theta-r)^{n-1}$ is $(2\cos\theta)^{n-1}$,

in $r^{n-2}(2\cos\theta-r)^{n-2}$ is $-(n-2)(2\cos\theta)^{n-3}$,

in $r^{n-3}(2\cos\theta-r)^{n-3}$ is $\dfrac{(n-3)(n-4)}{1.2}(2\cos\theta)^{n-5}$,

and so on. Hence

$$\frac{\sin n\theta}{\sin\theta}=(2\cos\theta)^{n-1}-(n-2)(2\cos\theta)^{n-3}+\frac{(n-3)(n-4)}{1.2}(2\cos\theta)^{n-5}-\ldots,$$

the general term being

$$(-1)^r\frac{(n-r-1)(n-r-2)\ldots(n-2r)}{r!}(2\cos\theta)^{n-2r-1},$$

and the last term being $(-)^{n/2-1}(n\cos\theta)$ or $(-)^{(n-1)/2}$ according as n is even or odd.

7·43. Expansion of cos $n\theta$ in a series of ascending powers of cos θ.

As in § 7·41, $\cos n\theta$ is the coefficient of r in $(1-r\cos\theta)(1-2r\cos\theta+r^2)^{-1}$, i.e.,

$=$[the coefficient of r^n in $(1-2r\cos\theta+r^2)^{-1}$

$-$the coefficient of r^{n-1} in $\cos\theta\,(1-2r\cos\theta+r^2)^{-1}$].

But $(1-2r\cos\theta+r^2)^{-1}$

$$=\{1+r(r-2\cos\theta)\}^{-1}$$
$$=1-r(r-2\cos\theta)+r^2(r-2\cos\theta)^2-\ldots$$
$$+(-)^m r^m(r-2\cos\theta)^m+\ldots\,. \qquad (1)$$

Two cases arise.

CASE I. Let n be even.

The first term which will contribute to r^n will be that for which $m=\frac{1}{2}n$ and there will be contributions due to the other terms that follow.

Thus the coefficient of r^n

in $(-)^{n/2}r^{n/2}(r-2\cos\theta)^{n/2}$ is $(-)^{n/2}$,

in $(-)^{n/2+1}r^{n/2+1}(r-2\cos\theta)^{n/2+1}$ is

$$(-)^{n/2+1}\frac{(\frac{1}{2}n+1)(\frac{1}{2}n+1-1)}{1\,.\,2}(-2\cos\theta)^2,$$

in $(-)^{n/2+2}r^{n/2+2}(r-2\cos\theta)^{n/2+2}$ is

$$(-)^{n/2+2}\frac{(\frac{1}{2}n+2)(\frac{1}{2}n+2-1)(\frac{1}{2}n+2-2)(\frac{1}{2}n+2-3)}{1\,.\,2\,.\,3\,.\,4}$$
$$\times(-2\cos\theta)^4,$$

and so on.

This shows that the coefficient of r^n in $(1-2r\cos\theta+r^2)^{-1}$ is

$$(-)^{n/2}\left[1-\frac{n(n+2)}{1.2}\cos^2\theta\right.$$
$$\left.+\frac{(n+4)(n+2)n(n-2)}{1.2.3.4}\cos^2\theta-\ldots\right].$$

Again the coefficient of r^{n-1}
in $(-)^{n/2} r^{n/2}(r-2\cos\theta)^{n/2}$ is $(-)^{n/2} \cdot \frac{1}{2}n(-2\cos\theta)$
in $(-)^{n/2+1} r^{n/2+1}(r-2\cos\theta)^{n/2+1}$ is

$$(-)^{n/2+1}\frac{(\frac{1}{2}n+1)(\frac{1}{2}n+1-1)(\frac{1}{2}n+1-2)}{1\,.\,2\,.\,3}\times(-2\cos\theta)^3,$$

and so on. Hence the coefficient of r^{n-1}
in $\cos\theta\,(1-2r\cos\theta+r^2)^{-1}$ is

$$-(-)^{n/2}\left[n\cos^2\theta-\frac{(n+2)n(n-2)}{1\,.\,2\,.\,3}\cos^4\theta-\ldots\right].$$

If follows that the coefficient of r^n in $(1-r\cos\theta)\times(1-2r\cos\theta+r^2)^{-1}$ is

$$(-)^{n/2}\left[1-\frac{n(n+2)}{1\,.\,2}\cos^2\theta+\frac{(n+4)(n+2)n(n-2)}{1\,.\,2\,.\,3\,.\,4}\times\cos^4\theta-\ldots\right],$$

$$+(-)^{n/2}\left[n\cos^2\theta-\frac{(n+2)n(n-2)}{1\,.\,2\,.\,3}\cos^4\theta+\ldots\right]$$

$$=(-)^{n/2}\left[1-\frac{n^2}{2!}\cos^2\theta+\frac{n^2(n^2-2^2)}{4!}\cos^4\theta-\ldots\right],$$

giving that

$$(-)^{n/2}\cos n\theta=1-\frac{n^2}{2!}\cos^2\theta+\frac{n^2(n^2-2^2)}{4!}\cos\theta-\ldots+(-)^{n/2}2^{n-1}\cos^n\theta, \qquad (2)$$

when n is even.

CASE II. Let n be odd.

The first term in (1) which will contribute to r^n will be that for $m=\frac{1}{2}(n+1)$ and there will be contributions from other terms also.

Thus the coefficient of r^n

in $(-)^{(n+1)/2}r^{(n+1)/2}(r-2\cos\theta)^{(n+1)/2}$

$$\text{is } (-)^{(n+1)/2}\tfrac{1}{2}(n+1)(-2\cos\theta).$$

in $(-)^{(n+3)/2}r^{(n+3)/2}(r-2\cos\theta)^{(n+3)/2}$ is

$$(-)^{(n+3)/2}\frac{\frac{1}{2}(n+3)\,.\,\frac{1}{2}(n+1)\,.\,\frac{1}{2}(n-1)}{1\,.\,2\,.\,3}(-2\cos\theta)^3,$$

in $(-)^{(n+5)/2}r^{(n+5)/2}(r-2\cos\theta)^{(n+5)/2}$ is

$$(-)^{(n+5)/2}\frac{\frac{1}{2}(n+5)\,.\,\frac{1}{2}(n+3)\,.\,\frac{1}{2}(n+1)\,.\,\frac{1}{2}(n-1)\,.\,\frac{1}{2}(n-3)}{1\,.\,2\,.\,3\,.\,4\,.\,5}\times(-2\cos\theta)^5,$$

and so on.

Thus the coefficient of r^n in (1) is

$$(-)^{(n+1)/2}\,\tfrac{1}{2}(n+1)(-2\cos\theta)$$
$$+(-)^{(n+3)/2}\frac{\frac{1}{2}(n+3)\,.\,\frac{1}{2}(n+1)\,.\,\frac{1}{2}(n-1)}{1\,.\,2\,.\,3}(-2\cos\theta)^3+$$
$$(-)^{(n+5)/2}\frac{\frac{1}{2}(n+5)\,.\,\frac{1}{2}(n+3)\,.\,\frac{1}{2}(n+1)\,.\,\frac{1}{2}(n-1)\,.\,\frac{1}{2}(n-3)}{1\,.\,2\,.\,3\,.\,4\,.\,5}\times(-2\cos\theta)^5+\ldots\,. \qquad (3)$$

The first term giving r^{n-1} in (1) will be from the term for which $m=\frac{1}{2}(n-1)$, so that proceeding as before, the coefficient of r^{n-1} in (1) is

$$(-)^{(n-1)/2}+(-)^{(n+1)/2}\frac{\frac{1}{2}(n+1)\,.\,\frac{1}{2}(n-1)}{1\,.\,2}(-2\cos\theta)^2$$
$$+(-)^{(n+3)/2}\frac{\frac{1}{2}(n+3)\,.\,\frac{1}{2}(n+1)\,.\,\frac{1}{2}(n-1)\,.\,\frac{1}{2}(n-3)}{1\,.\,2\,.\,3\,.\,4}\times(-2\cos\theta)+\ldots\,.$$

If we multiply this by $-\cos\theta$ and add to (3), we find that the coefficient of r^n in

$$(1-r\cos\theta)(1-2r\cos\theta+r^2)^{-1}$$

$$\text{is } (-)^{(n-1)/2} \cos\theta\,\{n+1-1\}$$
$$+(-)^{(n+1)/2}\frac{\cos^3\theta}{3!}\{n+3-3\}(n^2-1^2)$$
$$+(-)^{(n+3)/2}\frac{\cos^5\theta}{5!}\{n+5-5\}(n^2-1^2)(n^2-3^2)+\ldots$$
$$=(-)^{(n-1)/2}\Big[n\cos\theta-\frac{n(n^2-1^2)}{3!}\cos^3\theta$$
$$+\frac{n(n^2-1^2)(n^2-3^2)}{5!}\cos^5\theta-\ldots\Big].$$

It follows that, when n is odd,

$$(-)^{(n-1)/2}\cos n\theta = n\cos\theta-\frac{n(n^2-1^2)}{3!}\cos^3\theta$$
$$+\frac{n(n^2-1^2)(n^2-3^2)}{5!}\cos^5\theta-\ldots$$
$$+(-)^{(n-1)/2}2^{n-1}\cos^n\theta. \qquad (4)$$

7·44. Expansion of sin nθ/sin θ in a series of ascending powers of cos θ.

As in § 7·42, $\sin n\theta/\sin\theta$ is the coefficient of r^{n-1} in $(1-2r\cos\theta+r^2)^{-1}$,

i.e., in $\qquad \{1+r(r-2\cos\theta)\}^{-1}$,

i.e., in $1-r(r-2\cos\theta)+r^2(r-2\cos\theta)^2-\ldots$
$$+(-)^m r^m(r-2\cos\theta)^m+\ldots. \qquad (1)$$

We have to consider two cases.

CASE I. Let n be even.

The terms in (1) contributing to r^{n-1} will be those for which $m=\frac{1}{2}n,\ \frac{1}{2}n+1,\ \frac{1}{2}n+2$, etc. We see here that the coefficient of r^{n-1}

in $(-)^{n/2}r^{n/2}(r-2\cos\theta)^{n/2}$ is $(-)^{n/2}\cdot\frac{1}{2}n(2\cos\theta)$,

in $(-)^{n/2+1}r^{n/2+1}(r-2\cos\theta)^{n/2+1}$ is

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$$(-)^{n/2+1}\frac{(\frac{1}{2}n+1)(\frac{1}{2}n)(\frac{1}{2}n-1)}{3!}(-2\cos\theta)^3,$$

in $(-)^{n/2+2}r^{n/2+2}(r-2\cos\theta)^{n/2+2}$ is

$$(-)^{n/2+2}\frac{(\frac{1}{2}n+2)(\frac{1}{2}n+1)(\frac{1}{2}n)(\frac{1}{2}n-1)(\frac{1}{2}n-2)}{5!} \times(-2\cos\theta)^5,$$

and so on.

Adding these, we find that, when n is even,

$$(-)^{n/2+1}\frac{\sin n\theta}{\sin\theta}=n\cos\theta-\frac{n(n^2-2^2)}{3!}\cos^3\theta +\frac{n(n^2-2^2)(n^2-4^2)}{5!}\cos^5\theta-\dots +(-)^{n/2+1}(2\cos\theta)^{n-1}. \quad (2)$$

Case II. Let n be odd.

Here the first term that contributes to r^{n-1} is that given by $m=\frac{1}{2}(n-1)$. We have therefore to add up the coefficients from this and the succeeding terms. We find this sum to be equal to

$$(-)^{(n-1)/2}+(-)^{(n+1)/2}\frac{\frac{1}{2}(n+1)\frac{1}{2}(n-1)}{2!}(-\cos\theta)^2 +(-)^{(n+3)/2}\frac{\frac{1}{2}(n+3)\frac{1}{2}(n+1)\frac{1}{2}(n-1)\frac{1}{2}(n-3)}{4!} \times(-2\cos\theta)^4+\dots+(-2\cos\theta)^{n-1}.$$

Hence, when n is odd,

$$(-)^{(n-1)/2}\frac{\sin n\theta}{\sin\theta}=1-\frac{n^2-1^2}{2!}\cos^2\theta +\frac{(n^2-1^2)(n^2-3^2)}{4!}\cos^4\theta-\dots+(-)^{(n-1)/2}(2\cos\theta)^{n-1}. \quad (3)$$

7·45. Note. It is easily seen that the series deduced in § 7·43 and § 7·44 are the series of § 7·41 and § 7·42 respectively written backwards.

7·46. Expansions in series of powers of sin θ.

If we put $\pi-\theta$ for θ in (2) and (4) of § 7·43, and note that

$$\cos(\tfrac{1}{2}\pi-\theta)=\sin\theta,$$

$$\cos n(\tfrac{1}{2}\pi-\theta)=(-)^{n/2}\cos n\theta,\ n \text{ even}$$

and $$\cos n(\tfrac{1}{2}\pi-\theta)=(-)^{(n-1)/2}\sin n\theta,\ n \text{ odd},$$

we obtain that, when n is even,

$$\cos n\theta=1-\frac{n^2}{2!}\sin^2\theta+\frac{n^2(n^2-2^2)}{4!}\sin^4\theta-\dots$$
$$+(-)^{n/2}2^{n-1}\sin^n\theta, \quad (1)$$

and that, when n is odd,

$$\sin n\theta=n\sin\theta-\frac{n(n^2-1^2)}{3!}\sin^3\theta+\frac{n(n^2-1^2)(n^2-3^3)}{5!}$$
$$\times\sin^5\theta-\dots+(-)^{(n-1)/2}\,2^{n-1}\sin^n\theta. \quad (2)$$

Similarly, putting $\frac{1}{2}\pi-\theta$ for θ in (2) and (3) of § 7·44, we have, since

$$\sin n(\tfrac{1}{2}\pi-\theta)=(-)^{n/2+1}\sin n\theta,\ n \text{ even},$$

and $$\sin n(\tfrac{1}{2}\pi-\theta)=(-)^{(n-1)/2}\cos n\theta,\ n \text{ odd},$$

that when n is even

$$\frac{\sin n\theta}{\cos\theta}=n\sin\theta-\frac{n(n^2-2^2)}{3!}\sin^3\theta+\frac{n(n^2-2^2)(n^2-4^2)}{5!}$$
$$\times\sin^5\theta-\dots+(-)^{n/2+1}\,(2\sin\theta)^{n-1}, \quad (3)$$

and when n is odd,

$$\cos n\theta=\cos\theta\Big\{\ 1-\frac{n^2-1^2}{2!}\sin^2\theta+\frac{(n^2-1^2)(n^2-3^2)}{4!}$$

$$\times \sin^4 \theta - \ldots + (-)^{(n-1)/2} (2 \sin \theta)^{n-1}\Big\}. \qquad (4)$$

Ex. 1. Find the value of
$\cos \theta \cos (\theta + 2\pi/n) \cos (\theta + 4\pi/n) \ldots \cos \{\theta + (n-1)2\pi/n\}$.

We have seen that (§7·43)

$$nc - \frac{n(n^2-1^2)}{3!} c^3 + \frac{n(n^2-1^2)(n^2-3^2)}{5!} c^5 - \ldots + (-)^{(n-1)/2}\, 2^{n-1} c^n$$
$$= (-)^{(n-1)/2} \cos n\theta$$

and

$$1 - \frac{n^2}{2!} c^2 + \frac{n^2(n^2-2^2)}{4!} c^4 - \ldots + (-)^{n/2}\, 2^{n-1} c^n = (-)^{n/2} \cos n\theta,$$

according as n is odd or even, where $c \equiv \cos \theta$.

These give $\cos \theta$ when $\cos n\theta$ is given. But

$$\cos n\theta = \cos (n\theta + 2\pi) = \cos (n\theta + 4\pi) = \ldots,$$

i.e., $\cos n\theta$ remains unchanged when θ is increased by a multiple of $2\pi/n$. Hence these equations would also give

$$\cos \theta,\ \cos (\theta + 2\pi/n),\ \cos (\theta + 4\pi/n),\ \ldots \cos \{\theta + (n-1)2\pi/n\},$$

i.e., $\cos (\theta + 2\pi r/n)$, where $r = 0, 1, \ldots, n-1$.

For other values of r, the angles are repeated.

From the theory of equations,

$\cos \theta \cos (\theta + 2\pi/n) \cos (\theta + 4\pi/n) \ldots \cos \{\theta + (n-1)2\pi/n\}$
= product of the roots, and therefore it is equal to

$$\frac{(-)^{(n-1)/2} \cos n\theta}{(-)^{(n-1)/2}\, 2^{n-1}}, \text{ i.e., } \cos n\theta/2^{n-1}, \text{ if } n \text{ be odd,}$$

and $\dfrac{1 - (-)^{n/2} \cos n\theta}{(-)^{n/2}\, 2^{n-1}}$, i.e., $(\frac{1}{2})^{n-1}[(-)^{n/2} - \cos n\theta)$, if n be even.

Ex. 2. Prove that

$$\sin^{-1} x = x + \left(\frac{1}{2}\right)\frac{x^3}{3} + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)\frac{x^5}{5} + \ldots.$$

We have seen that

$$\sin n\theta = n \sin \theta - \frac{n(n^2-1^2)}{3!} \sin^3 \theta + \frac{n(n^2-1^2)(n^2-3^2)}{5!} \sin^5 \theta - \ldots.$$

Also $$\sin n\theta = n\theta - \frac{n^3\theta^3}{3!} + \frac{n^5\theta^5}{5!} - \dots .$$

Equating the coefficients of n in the two expansions of $\sin n\theta$, we have

$$\theta = \sin\theta + \frac{1}{3!}\sin^3\theta + \frac{1^2 . 3^2}{5!}\sin^5\theta + \dots$$
$$= \sin\theta + \tfrac{1}{2} . \frac{\sin^3\theta}{3} + \frac{1 . 3}{2 . 4}\ \frac{\sin^5\theta}{5} + \dots .$$

Put $\sin\theta = x$, so that $\theta = \sin^{-1} x$.

Then $$\sin^{-1} x = x + \left(\frac{1}{2}\right)\frac{x^3}{3} + \left(\frac{1 . 3}{2 . 4}\right)\frac{x^5}{5} + \dots .$$

EXAMPLES

Prove that

1. $\sin 4\theta = \sin\theta[(2\cos\theta)^3 - 4\cos\theta]$.

2. $\sin 5\theta = \sin\theta[(2\cos\theta)^4 - 12\cos^2\theta + 1]$
$= 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$.

3. $2\cos 6\theta = (2\cos\theta)^6 - 96\cos^4\theta + 36\cos^2\theta - 2$.

4. $2\cos 7\theta = (2\cos\theta)^7 - 224\cos^5\theta + 112\cos^3\theta - 14\cos\theta$.

5. $\sin 8\theta = \cos\theta[8\sin\theta - 80\sin^3\theta + 192\sin^5\theta - 128\sin^7\theta]$. [*Luck.*, '63]

6. $\cos 9\theta = \cos\theta[1 - 40\sin^2\theta + 240\sin^4\theta - 448\sin^6\theta + 256\sin^8\theta]$.

7. Prove that
$$\frac{\cos 11\theta}{\cos\theta} = x^5 - x^4 - 4x^3 + 3x^2 + 3x - 1,$$
where $x = 2\cos 2\theta$.

8. Express $\cos 7\theta$ in terms of $\cos\theta$ and show that $\cos(\pi/7)$ is a root of the equation
$$8x^3 - 4x^2 - 4x + 1 = 0.$$
What are the other roots of the equation ? [*Andhra*, 1931]

9. Prove that
$$1 + \cos 10\theta = 2(16\cos^5\theta - 20\cos^3\theta + 5\cos\theta)^2.$$ [*Luck.*, '55]

10. Expand $\cos n\theta$ in descending powers of $\cos\theta$ and hence show that

$$\cos\frac{\pi}{4n}\cos\frac{3\pi}{4n}\cos\frac{5\pi}{4n}\ldots\cos\frac{2n-1}{4n}\pi=\frac{1}{2^{n-1/2}}.$$

Find the value of

11. $\sin\theta\sin(\theta+2\pi/n)\sin(\theta+4\pi/n)\ldots\sin\{\theta+(n-1)2\pi/n\}$.

12. $\sec\theta+\sec(\theta+2\pi/n)+\sec(\theta+4\pi/n)+\ldots$ to n terms.

13. $\operatorname{cosec}^2\theta+\operatorname{cosec}^2(\theta+2\pi/n)+\operatorname{cosec}^2(\theta+4\pi/n)$
$+\ldots$ to n terms.

14. $\sec^2\theta+\sec^2(\theta+2\pi/n)+\sec^2(\theta+4\pi/n)+\ldots$ to n terms.

15. Prove that

$$\frac{1}{2}\theta^2=\frac{1}{2}\sin^2\theta+\frac{2}{3}\cdot\frac{\sin^4\theta}{4}+\frac{2\,.\,4}{3\,.\,5}\cdot\frac{\sin^6\theta}{6}+\ldots.$$

16. Prove that

$$\tfrac{1}{6}(\sin^{-1}x)^3=\frac{1}{3!}x^3+\frac{1^2+3^2}{7!}x^5+\frac{1^2\,.\,3^2+3^2\,.\,5^2+5^2\,.\,1^2}{7!}x^7+\ldots.$$

17. Find the value of

$$1-\frac{n^2-1^2}{2!}\tfrac{1}{2}+\frac{(n^2-1^2)\,(n^2-3^2)}{4!}\cdot\frac{1}{2^2}-\ldots+(-2)^{(n-1)/2},$$

when n is an odd positive integer.

18. Prove that when n is odd

$$\sqrt{2}\cos\tfrac{1}{4}n\pi=(-)^{(n-1)/2}\Big[n-\frac{n(n^2-1^2)}{3!}(\tfrac{1}{2})$$
$$+\frac{n(n^2-1^2)(n^2-3^2)}{5!}(\tfrac{1}{4})-\ldots\Big].$$

[Hint. Put $\theta=\frac{1}{4}\pi$ in (4) of § 7·43.]

19. Expand $\cos^2 n\theta$ in ascending powers of $\cos\theta$.

20. Prove that

$$\sec\theta=1+\tfrac{1}{2}\sin^2\theta+\frac{1\,.\,3}{2\,.\,4}\sin^4\theta+\ldots.$$

Examples on Chapter VII

1. If

$$64\cos^3\theta\sin^4\theta=A\cos 7\theta+B\cos 5\theta+C\cos 3\theta+D\cos\theta,$$

find the values of the coefficients A, B, C, D. [*Lucknow*, 1956]

2. Prove that

$$2\cos n\theta = (2\cos\theta)^n - n(2\cos\theta)^{n-2} + \frac{n(n-3)}{2!}(2\cos\theta)^{n-4} - \dots$$

from the identity $p^n + q^n = (p+q)^n - n(p+q)^{n-2}pq$

$$+\frac{n(n-3)}{2!}(p+q)^{n-4}p^2q^2 - \dots$$ [*Travancore*, '40]

3. Express $\cos 8\theta$ in powers of $\cos\theta$ and express $\cos^8\theta$ as the sum of cosines of angles which are multiples of θ.

4. Prove the identity

$$\frac{(1-\cos 14\theta)(1-\cos\theta)}{(1-\cos 2\theta)(1-\cos 7\theta)} = (2\cos 3\theta - 2\cos 2\theta + 2\cos\theta - 1)^2.$$

5. Prove that

$$\theta\sec\theta = \sin\theta + \tfrac{2}{3}\sin^3\theta + \frac{2\,.\,4}{3\,.\,5}\sin^5\theta + \dots\,.$$

6. Prove that

$$\frac{\sin^{-1}x}{\sqrt{(1-x^2)}} = x + \tfrac{2}{3}x^3 + \frac{2\,.\,4}{3\,.\,5}x^5 + \dots\,.$$

[Hint. Put $\sin\theta = x$ in Q. 5.]

7. From the identity

$$\cos^n\theta(\cos n\theta - i\sin n\theta) = \frac{1}{(1+i\tan\theta)^n},$$

prove that

$$\cos^n\theta\cos n\theta = 1 - \frac{n(n+1)}{2}\tan^2\theta + \frac{n(n+1)(n+2)(n+3)}{4!}\tan^4\theta - \dots$$

and $$\cos^n\theta\sin n\theta = n\tan\theta - \frac{n(n+1)(n+2)}{3!}\tan^3\theta + \dots$$

8. Prove that

$$\pi^2 = 18\sum_{n=0}^{\infty}\frac{n!\,n!}{(2n+2)!}.$$ [*Alld.*, '53]

[Hint. Put $\theta = \pi/6$ in the result of Q. 15, p. 118.]

9. Prove that

$$\tan^{-1} y = \frac{y}{1+y^2}\left\{1+\frac{2}{3}\cdot\frac{y^2}{1+y^2}+\frac{2\cdot 4}{3\cdot 5}\left(\frac{y^2}{1+y^2}\right)^2+\dots\right\}.$$

[Hint. Put $\theta=\tan^{-1} y$ in Q. 5.]

10. Expand $\sin^2 n\theta$ in ascending powers of $\sin\theta$.

11. Find the sum of

$\tan^2\theta+\tan^2(\theta+2\pi/n)+\tan^2(\theta+4\pi/n)+\dots$ to n terms.

12. Find the sum of the products, taken two at a time, of expressions of the form $\sec(\theta+2r\pi/n)$, where r has all values from zero to $n-1$.

13. Prove that

$$\frac{2^6\sin^7\theta+\sin 7\theta}{2^6\cos^7\theta-\cos 7\theta}=\tan\theta\tan^2(\theta+\tfrac{1}{6}\pi)\tan^2(\theta-\tfrac{1}{6}\pi).$$

14. Show that $2^{12}\cos^{13}\theta-\cos 13\theta$ is divisible by

$$(1+2\cos 2\theta)^2.$$

15. Prove that if m be even

$$\frac{1+\cos(2m+1)\theta}{1+\cos\theta}=\left\{1+m\cos\theta-\frac{m(m+2)}{2!}\cos^2\theta-\frac{(m-2)\,(m)\,(m+2)}{3!}\cos^3\theta+\frac{(m-2)m(m+2)(m+4)}{4!}\cos^4\theta-\dots+(-)^{m/2}(2\cos\theta)^m\right\}^2.$$

CHAPTER VIII

EXPANSIONS OF TRIGONOMETRICAL FUNCTIONS (*contd.*)

8·1. Expansion of e^{ax} cos bx in a series of ascending powers of x. Using Euler's theorem, we have

$$e^{ax}\cos bx = e^{ax} \cdot \tfrac{1}{2}(e^{ibx}+e^{-ibx})$$
$$= \tfrac{1}{2}[e^{(a+ib)x}+e^{(a-ib)x}].$$

Expanding the exponential expressions on the right, we have

$$e^{ax}\cos bx = \tfrac{1}{2}\Big[\Big\{1+(a+ib)x+\frac{(a+ib)^2x^2}{2!}+\dots\Big\}$$
$$+\Big\{1+(a-ib)x+\frac{(a-ib)^2x^2}{2!}+\dots\Big\}\Big]$$
$$= \tfrac{1}{2}[2+x(a+ib+a-ib)+\frac{x^2}{2!}\{(a+bi)^2+$$
$$(a-ib)^2\}+\dots+\frac{x^n}{n!}\{(a+ib)^n+(a-ib)^n\}+\dots].$$

Put $a=r\cos\alpha$ and $b=r\sin\alpha$. Then the coefficient of x^n in the above expansion is

$$(1/2\,.\,n!)[\{r(\cos\alpha+i\sin\alpha)\}^n+\{r(\cos\alpha-i\sin\alpha)\}^n]$$
$$=(r^n/2\,.\,n!)[(\cos\alpha+i\sin\alpha)^n+(\cos\alpha-i\sin\alpha)^n]$$
$$=(r^n/2\,.\,n!)\,[\cos n\alpha+i\sin n\alpha+\cos n\alpha-i\sin n\alpha),$$

by De Moivre's Theorem

$$=(r^n/n!)\cos n\alpha$$
$$=(a^2+b^2)^{n/2}(1/n!)\cos\{n\tan^{-1}(b/a)\},$$

for $r^2=a^2+b^2$ and $\tan\alpha=b/a$.

Hence

$$e^{ax}\cos bx=\sum_{n=0}^{\infty}\frac{x^n}{n!}(a^2+b^2)^{n/2}\cos\left(n\tan^{-1}\frac{b}{a}\right).$$

8·2. When sin x = n sin (x+α). *If* $\sin x = n\sin(x+\alpha)$, *to expand x in a series of ascending powers of n, where n is less than unity.*

Here

$$e^{ix}-e^{-ix}=n\{e^{i(x+\alpha)}-e^{-i(x+\alpha)}\},$$

$\therefore$ $$e^{2ix}-1=n(e^{2ix}e^{i\alpha}-e^{-i\alpha}),$$

or $$e^{2ix}(1-ne^{i\alpha})=1-ne^{-i\alpha},$$

or $$e^{2ix}=\frac{1-ne^{-i\alpha}}{1-ne^{i\alpha}}.$$

Taking the logarithms of the two sides, we have, since $|n|<1$,

$$\begin{aligned}2ix&=\log(1-ne^{-i\alpha})-\log(1-ne^{i\alpha})\\&=-ne^{-i\alpha}-\tfrac{1}{2}n^2e^{-2i\alpha}-\tfrac{1}{3}n^3e^{-3i\alpha}-\dots\\&\quad+ne^{i\alpha}+\tfrac{1}{2}n^2e^{2i\alpha}+\tfrac{1}{3}n^3e^{3i\alpha}+\dots\\&=n(e^{i\alpha}-e^{-i\alpha})+\tfrac{1}{2}n^2(e^{2i\alpha}-e^{-2i\alpha})\\&\qquad+\tfrac{1}{3}n^3(e^{3i\alpha}-e^{-3i\alpha})+\dots.\end{aligned}$$

$$=n(2i\sin\alpha)+\tfrac{1}{2}n^2(2i\sin 2\alpha)+\tfrac{1}{3}n^3(2i\sin 3\alpha)+\dots$$

$\therefore$ $$x=n\sin\alpha+\tfrac{1}{2}n^2\sin 2\alpha+\tfrac{1}{3}n^3\sin 3\alpha+\dots. \qquad (1)$$

It is assumed here that x lies between $\pi/2$ and $-\pi/2$. If not, we take $\log_e 2ix$ to be equal to $2xi+2k\pi i$. The right hand side of (1) will then be equal to $x+k\pi$, where k is such that $x+k\pi$ lies between $\pi/2$ and $-\pi/2$.

8·3. When tan x = n tan y. *If* $\tan x = n\tan y$, *to find a series for x.* [*Lucknow*, '63]

Here

$$\frac{e^{xi}-e^{-xi}}{e^{xi}+e^{-xi}}=n\frac{e^{iy}-e^{-iy}}{e^{iy}+e^{-iy}},$$

or

$$\frac{e^{2xi}-1}{e^{2xi}+1}=n\frac{e^{2iy}-1}{e^{2iy}+1}.$$

Therefore

$$e^{2xi}=\frac{(1+n)e^{2iy}+1-n}{(1-n)e^{2iy}+1+n}$$

$$=e^{2iy}\frac{1+me^{-2iy}}{1+me^{2iy}},\quad\text{where } m=\frac{1-n}{1+n}.$$

Taking the logarithms of the two sides, we have

$$2xi=2yi+\log(1+me^{-2iy})-\log(1+me^{2iy})$$

$$=2yi-m(e^{2iy}-e^{-2iy})+\tfrac{1}{2}m^2(e^{4iy}-e^{-4iy})-\ldots$$

Hence

$$x=y-m\sin 2y+\tfrac{1}{2}m^2\sin 4y-\tfrac{1}{3}m^3\sin 6y+\ldots .$$

Ex. 1. Prove that

$$\frac{(1-x^2)\cos\theta}{1-2x^2\cos 2\theta+x^4}=\cos\theta+x^2\cos 3\theta+x^4\cos 5\theta+\ldots,$$

when $-1<x<1$.

Writing the expenential values of $\cos\theta$ and $\cos 2\theta$, we have

$$\frac{(1-x^2)\cos\theta}{1-2x^2\cos 2\theta+x^4}=\frac{(1-x^2)(e^{\theta i}+e^{-\theta i})}{2[1-x^2e^{2\theta i}-x^2e^{-2\theta i}+x^4]}$$

$$=\frac{(1-x^2)(e^{\theta i}+e^{-\theta i})}{2(1-x^2e^{2\theta i})(1-x^2e^{-2\theta i})}$$

$$=\tfrac{1}{2}\left[\frac{e^{\theta i}}{1-x^2e^{2\theta i}}+\frac{e^{-\theta i}}{1-x^2e^{-2\theta i}}\right]$$

$$=\tfrac{1}{2}[e^{\theta i}(1-x^2e^{2\theta i})^{-1}+e^{-\theta i}(1-x^2e^{-2\theta i})^{-1}]$$

$$=\tfrac{1}{2}[e^{\theta i}(1+x^{2\theta}e^{2\theta i}+x^4e^{4\theta i}+x^6e^{6\theta i}+\ldots)$$

$$+e^{-\theta i}(1+x^2e^{-2\theta i}+x^{4\theta}e^{-4\theta i}+\ldots)]$$

$$=\tfrac{1}{2}(e^{\theta i}+e^{-\theta i})+x^2\cdot\tfrac{1}{2}(e^{3\theta i}+e^{-3\theta i})$$
$$+x^4\cdot\tfrac{1}{2}(e^{5\theta i}+e^{-5\theta i})+\ldots$$
$$=\cos\theta+x^2\cos 3\theta+x^4\cos 5\theta+\ldots.$$

Ex. 2. Expand $\log(1+x+x^2)$ in powers of x when x is small. [*Annam.*, '34]

Solving $x^2+x+1=0$, we have

$$x=-\tfrac{1}{2}+i\frac{\sqrt{3}}{2} \text{ or } x=-\tfrac{1}{2}-i\frac{\sqrt{3}}{2}$$

i.e.,
$$-\left(\tfrac{1}{2}+i\frac{\sqrt{3}}{2}\right)x=1 \text{ or } -\left(\tfrac{1}{2}-i\frac{\sqrt{3}}{2}\right)x=1.$$

Therefore

$$1+x+x^2=\left[1+\left(\tfrac{1}{2}+i\frac{\sqrt{3}}{2}\right)x\right]\left[1+\left(\tfrac{1}{2}-i\frac{\sqrt{3}}{2}\right)x\right],$$

whence

$$\log(1+x+x^2)=\log\left\{1+\left(\tfrac{1}{2}+i\frac{\sqrt{3}}{2}\right)x\right\}$$
$$+\log\left\{1+\left(\tfrac{1}{2}-i\frac{\sqrt{3}}{2}\right)x\right\}$$
$$=\left[\left(\tfrac{1}{2}+i\frac{\sqrt{3}}{2}\right)x-\tfrac{1}{2}\left(\tfrac{1}{2}+i\frac{\sqrt{3}}{2}\right)^2x^2+\tfrac{1}{3}\left(\tfrac{1}{2}+i\frac{\sqrt{3}}{2}\right)^3x^3-\ldots\right.$$
$$\left.+(-1)^{n-1}\frac{1}{n}\left(\tfrac{1}{2}+i\frac{\sqrt{3}}{2}\right)^nx^n+\ldots\right]$$
$$+\left[\left(\tfrac{1}{2}-i\frac{\sqrt{3}}{2}\right)x-\tfrac{1}{2}\left(\tfrac{1}{2}-i\frac{\sqrt{3}}{2}\right)^2x^2+\ldots\right.$$
$$\left.+(-1)^{n-1}\frac{1}{n}\left(\tfrac{1}{2}-i\frac{\sqrt{3}}{2}\right)^nx^n+\ldots\right]$$
$$=\left[\left(\tfrac{1}{2}+i\frac{\sqrt{3}}{2}\right)+\left(\tfrac{1}{2}-i\frac{\sqrt{3}}{2}\right)\right]x-\tfrac{1}{2}\left[\left(\tfrac{1}{2}+i\frac{\sqrt{3}}{2}\right)^2+\left(\tfrac{1}{2}-i\frac{\sqrt{3}}{2}\right)^2\right]x^2$$
$$+\ldots+(-1)^{n-1}\frac{1}{n}\left[\left(\tfrac{1}{2}+i\frac{\sqrt{3}}{2}\right)^n+\left(\tfrac{1}{2}-i\frac{\sqrt{3}}{2}\right)^n\right]x^n+\ldots.$$

The coefficient of x^n in the above expansion is

$$(-1)^{n-1}\frac{1}{n}\left[\left(\tfrac{1}{2}+i\frac{\sqrt{3}}{2}\right)^n+\left(\tfrac{1}{2}-i\frac{\sqrt{3}}{2}\right)^n\right]$$

$$=(-1)^{n-1}\,\frac{1}{n}\left[\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)^n+\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)^n\right]$$

$$=(-1)^{n-1}\,\frac{1}{n}\left[\cos\frac{n\pi}{3}+i\sin\frac{n\pi}{3}+\cos\frac{n\pi}{3}-i\sin\frac{n\pi}{3}\right]$$

$$=(-1)^{n-1}\,\frac{1}{n}\left[2\cos\frac{n\pi}{3}\right]$$

$$=(-1)^{n-1}\,\frac{2}{n}\cos\frac{n\pi}{3}.$$

Hence

$$\log(1+x+x^2)=\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{2}{n}\cos\frac{n\pi}{3}\cdot x^n.$$

Ex. 3. If a, b and c be the sides of a triangle, C the angle opposite to c and $b<a$, show that

$$\log c=\log a-\frac{b}{a}\cos C-\frac{b^2}{2a^2}\cos 2C-\frac{b^3}{3a^3}\cos 3C-\ldots$$

[*Lucknow*, '61]

In a triangle, we have

$$c^2=a^2+b^2-2ab\cos C.$$

Hence

$$c^2=a^2+b^2-ab(e^{Ci}+e^{-Ci})$$

$$=a^2\left(1-\frac{b}{a}e^{Ci}\right)\left(1-\frac{b}{a}e^{-Ci}\right).$$

Thus

$$2\log c=2\log a+\log\left(1-\frac{b}{a}e^{Ci}\right)+\log\left(1-\frac{b}{a}e^{-Ci}\right)$$

$$=2\log a-\frac{b}{a}(e^{Ci}+e^{-Ci})-\frac{b^2}{2a^2}(e^{2Ci}+e^{-2Ci})-\ldots$$

$$\therefore\quad \log c=\log a-\frac{b}{a}\cos C-\frac{b^2}{2a^2}\cos 2C-\ldots.$$

This gives a series for the logarithm of the third side, when two sides of a triangle and the included angle are given.

EXAMPLES

1. Expand $e^{ax}\sin bx$ in ascending powers of x.

2. If $\sin\theta=x\cos(\theta+\alpha)$, where $|x|<1$, expand θ in a series of ascending powers of x. [*Lucknow*, '63]

3. If $\tan \frac{1}{2}\theta = 2 \tan \frac{1}{2}\alpha$, and θ, α are acute, prove that

$$\tfrac{1}{2}(\theta-\alpha)=\tfrac{1}{3}\sin\alpha+\frac{1}{2\,.\,3^2}\sin 2\alpha+\frac{1}{3\cdot 3^3}\sin 3\alpha+\ldots$$

4. If $\cot y = \cot x + \operatorname{cosec}\alpha \operatorname{cosec} x$, show that

$$y=\sin\alpha\sin x-\frac{\sin^2\alpha}{2}\sin 2x+\frac{\sin^3\alpha}{3}\sin 3x-\ldots .$$

5. Prove that, if $x<1$,

$$\frac{1-x^2}{1-2x\cos\theta+x^2}=1+\sum_{n=1}^{\infty}(2x^n\cos n\theta). \qquad \text{[\textit{Banaras}, 1951]}$$

6. Prove that, if $x<1$,

$$\frac{x\sin\theta}{1-2x\cos\theta+x^2}=x\sin\theta+x^2\sin 2\theta+x^3\sin 3\theta+\ldots .$$

[*Lucknow*, '57]

7. Show that

$$\frac{1}{1+e\cos\alpha}=\frac{1}{\sqrt{(1-e^2)}}(1-2m\cos\alpha+2m^2\cos 2\alpha-\ldots),$$

where $e(1+m^2)=2m$.

8. If $-1<x<1$, expand $\log(1-2x\cos\theta+x^2)$ in a series of cosines of multiples of θ. [*Agra*, 1945]

9. Expand in an infinite series

$$e^{a\cos\phi}\cos(\theta+a\sin\phi). \qquad \text{[\textit{Agra}, 1951]}$$

10. Prove that, if $a<1$,

$$\tan^{-1}\frac{a\sin\theta}{1-a\cos\theta}=a\sin\theta+\tfrac{1}{2}a^2\sin 2\theta+\tfrac{1}{3}a^3\sin 3\theta+\ldots .$$

[*Banaras*, '63]

11. If $|x|<1$, expand

$$\tan^{-1}\frac{2x\cos\theta}{1-x^2}$$

in ascending powers of x.

12. Show that, if $|x|<1$,

$$\frac{\cos\phi-x\cos(\theta+\phi)}{1-2x\cos\theta+x^2}=\cos\phi+x\cos(\theta-\phi)+x^2\cos(2\theta-\phi)+\ldots .$$

13. If in a triangle ABC, B is less than A, prove that

$$B = \frac{b}{a}\sin C + \frac{b^2}{2a^2}\sin 2C + \frac{b^3}{3a^3}\sin 3C + \dots .$$

14. If a and b be the sides of a plane triangle, and A and B the opposite angles, then show that

$$\log b - \log a = (\cos 2A - \cos 2B) + \tfrac{1}{2}(\cos 4A - \cos 4B) + \tfrac{1}{3}(\cos 6A - \cos 6B) + \dots$$

[*Luck.*, '65]

8·4. Gregory's series. *To show that, if*

$$-\tfrac{1}{4}\pi \leqslant \theta \leqslant \tfrac{1}{4}\pi,$$

$$\theta = \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \dots ad\ inf.$$

We have

$$1 + i\tan\theta = \sec\theta(\cos\theta + i\sin\theta) = \sec\theta \,.\, e^{i\theta}.$$

Therefore

$$\log\sec\theta + i\theta = \log(1 + i\tan\theta) = i\tan\theta + \tfrac{1}{2}\tan^2\theta - \tfrac{1}{3}i\tan^3\theta - \tfrac{1}{4}\tan^4\theta + \tfrac{1}{5}i\tan^5\theta - \dots \text{ad inf.},$$

when $\tan\theta$ is numerically less than or equal to 1, that is, when $-\frac{1}{4}\pi \leqslant \theta \leqslant \frac{1}{4}\pi$.

Equating the imaginary parts on each side of this equation, we have

$$\theta = \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \dots \text{ad inf.}$$

This is called Gregory's series.*

If we put $\tan\theta = x$, that is, $\theta = \tan^{-1}x$, we have Gregory's series in another form, viz.

$$\tan^{-1}x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \dots,$$

true when $|x| \leqslant 1$.

*This series was known in India in the 15th century A.D. But in Europe it was first given by James Gregory (1638-1675), Professor at Edinburgh, and so it is now called after his name.

The general term of this series is

$$(-)^{n-1} x^{2n-1}/(2n-1),$$

and thus we have

$$\tan^{-1} x = \sum_{n=1}^{\infty} \{(-)^{n-1} x^{2n-1}/(2n-1)\}.$$

8·5. General Value. In case θ does not lie between $-\pi/4$ and $\pi/4$, suppose that $\theta = n\pi + \phi$, where ϕ lies between $-\pi/4$ and $\pi/4$. Then

$$\phi = \tan\phi - \tfrac{1}{3}\tan^2\phi + \tfrac{1}{5}\tan^5\phi - \dots$$

that is,

$$\theta - n\pi = \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \dots,$$

true when

$$n\pi - \tfrac{1}{4}\pi \leqslant \theta \leqslant n\pi + \tfrac{1}{4}\pi.$$

As a particular case of this general result, suppose that θ lies between $\frac{3}{4}\pi$ and $\frac{5}{4}\pi$ then

$$\theta - \pi = \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \dots .$$

If θ lies between $\frac{11}{4}\pi$ and $\frac{13}{4}\pi$, we have

$$\theta - 3\pi = \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \dots .$$

Similarly, if θ lies between $-\frac{9}{4}\pi$ and $-\frac{7}{4}\pi$, the corresponding expansion is

$$\theta + 2\pi = \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \dots .$$

8·6. Evaluation of π. Gregory's series has been used very extensively for the calculation of the value of π correct to various decimal places. We give below the important ones.

(i) GREGORY'S SERIES. If we put $\theta = \pi/4$ in Gregory's series we have

$$\tfrac{1}{4}\pi = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} - \dots .$$

This series is not rapidly convergent and hence a large number of terms will have to be taken in order to obtain the value of π correct to a certain decimal place.

(ii) EULER'S SERIES. It is easily seen that

$$\tan^{-1}\tfrac{1}{2}+\tan^{-1}\tfrac{1}{3}=\tfrac{1}{4}\pi.$$

Expanding each of these inverse tangent functions, we have

$$\tfrac{1}{4}\pi=\tfrac{1}{2}-\tfrac{1}{3}\left(\frac{1}{2^3}\right)+\tfrac{1}{5}\left(\frac{1}{2^5}\right)-\dots$$

$$+\tfrac{1}{3}-\tfrac{1}{3}\left(\frac{1}{3^3}\right)+\tfrac{1}{5}\left(\frac{1}{3^5}\right)-\dots.$$

Hence

$$\tfrac{1}{4}\pi=(\tfrac{1}{2}+\tfrac{1}{3})-\tfrac{1}{3}\left(\frac{1}{2^3}+\frac{1}{3^3}\right)+\tfrac{1}{5}\left(\frac{1}{2^5}+\frac{1}{3^5}\right)-\dots.$$

(iii) MACHIN'S SERIES. This is obtained by using the identity

$$\tfrac{1}{4}\pi=4\tan^{-1}\tfrac{1}{5}-\tan^{-1}\tfrac{1}{239}.$$

We can establish this identity thus:

$$2\tan^{-1}\tfrac{1}{5}=\tan^{-1}\{\tfrac{2}{5}/(1-\tfrac{1}{25})\}=\tan^{-1}\tfrac{5}{12},$$

and

$$4\tan^{-1}\tfrac{1}{5}=2\tan^{-1}\tfrac{5}{12}=\tan^{-1}\{\tfrac{10}{12}/(1-\tfrac{25}{144})\}$$
$$=\tan^{-1}\tfrac{120}{119}.$$

Since $4\tan^{-1}\frac{1}{5}$ is a little greater than $\pi/4$, suppose that

$$4\tan^{-1}\tfrac{1}{5}=(\tfrac{1}{4}\pi+\tan^{-1}x).$$

Hence

$$\tan(4\tan^{-1}\tfrac{1}{5})=\tfrac{120}{119}=\tan(\tfrac{1}{4}\pi+\tan^{-1}x)$$
$$=(1+x)/(1-x),$$

giving $x=\frac{1}{239}$. This proves the above identity.

It follows from Gregory's series that

$$\tfrac{1}{4}\pi = 4\left\{\frac{1}{5} - \frac{1}{3}\cdot\frac{1}{5^3} + \frac{1}{5}\cdot\frac{1}{5^5} - \frac{1}{7\cdot 5^7} + \dots\right\}$$
$$-\left\{\frac{1}{239} - \frac{1}{3}\cdot\frac{1}{(239)^3} + \frac{1}{5}\cdot\frac{1}{(239)^5} - \dots\right\}.$$

(iv) RUTHERFORD'S SERIES. It is easy to see that

$$\tan^{-1}\tfrac{1}{239} = \tan^{-1}\tfrac{1}{70} - \tan^{-1}\tfrac{1}{99}.$$

Hence, from the identity in (iii),

$$\tfrac{1}{4}\pi = 4\tan^{-1}\tfrac{1}{5} - \tan^{-1}\tfrac{1}{70} + \tan^{-1}\tfrac{1}{99}$$
$$= 4\left(\frac{1}{5} - \frac{1}{3.5^3} + \dots\right)$$
$$-\left(\frac{1}{70} - \frac{1}{3.70^3} + \dots\right) + \left(\frac{1}{99} - \frac{1}{3.99^3} + \dots\right).$$

NOTE. Rutherford's series is more rapidly convergent than Machin's series and Machin's series is more rapidly convergent than Euler's series. The value of π has been calculated up to 707 places of decimals.

Ex. Show that

$$\tfrac{1}{4}\pi = \left(\frac{2}{3} + \frac{1}{7}\right) - \frac{1}{3}\left(\frac{2}{3^3} + \frac{1}{7^3}\right) + \frac{1}{5}\left(\frac{2}{3^5} + \frac{1}{7^5}\right) - \dots .$$

[*Travan.*, '51; *Alld.*, '54]

The given series

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n-1}\left\{2.\frac{1}{3^{2n-1}} + \frac{1}{7^{2n-1}}\right\}$$
$$= 2\tan^{-1}(\tfrac{1}{3}) + \tan^{-1}(\tfrac{1}{7})$$
$$= \tan^{-1}(\tfrac{3}{4}) + \tan^{-1}(\tfrac{1}{7})$$
$$= \tan^{-1}(1) \text{ or } \pi/4.$$

Hence the result.

Examples

1. Prove that

$$\tfrac{1}{8}\pi = \frac{1}{1.3} + \frac{1}{5.7} + \frac{1}{9.11} + \dots .$$

2. For the validity of

$$\theta - n\pi = \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \dots,$$

write down the values of n, when θ lies between

$\tfrac{7}{4}\pi$ and $\tfrac{9}{4}\pi$, $-\tfrac{5}{4}\pi$ and $-\tfrac{3}{4}\pi$, and $\tfrac{19}{4}\pi$ and $\tfrac{21}{4}\pi$.

3. Prove that

$$\pi = 2\sqrt{3}\left\{1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \dots\right\}.$$

[*Andhra*, '50; *Gauhati*, '55]

4. If $x > 0$, prove that

$$\tan^{-1} x = \frac{\pi}{4} + \frac{x-1}{x+1} - \frac{1}{3}\left(\frac{x-1}{x+1}\right)^3 + \frac{1}{5}\left(\frac{x-1}{x+1}\right)^5 - \dots .$$

[*Travancore*, '44]

5. If $\theta < \dfrac{\pi}{4}$, prove that

$$\log\sec\theta = \tfrac{1}{2}\tan^2\theta - \tfrac{1}{4}\tan^4\theta + \tfrac{1}{6}\tan^6\theta - \dots .$$ [*Cal.*, 1946]

6. If $-\dfrac{\pi}{4} \leqslant \theta \leqslant \dfrac{\pi}{4}$, expand $\tan^{-1}\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$ as a power series in $\tan\theta$.

7. If $0 < \theta < \tfrac{1}{2}\pi$, prove that

$$\tan^{-1}\frac{1-\cos\theta}{1+\cos\theta} = \tan^2\tfrac{1}{2}\theta - \tfrac{1}{3}\tan^6\tfrac{1}{2}\theta + \tfrac{1}{5}\tan^{10}\tfrac{1}{2}\theta - \dots .$$

8. Show that

$$\tanh\theta + \tfrac{1}{3}\tanh^3\theta + \tfrac{1}{5}\tanh^5\theta + \dots$$
$$= \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \dots$$

Examples on Chapter VIII

1. Prove that

$$\log\frac{a^2}{a^2\cos^2\theta + b^2\sin^2\theta} = 4[c\sin^2\theta - \tfrac{1}{2}c^2\sin^2 2\theta + \tfrac{1}{3}c^3\sin^2 3\theta - \dots],$$

where $c = (a-b)/(a+b)$.

2. Expand in an infinite series

$$\frac{\cos\theta-a\cos(\theta-\phi)}{1-2a\cos\phi+a^2},$$

where $|a|<1$.

3. If θ be a positive acute angle, expand $\log\tan(\frac{1}{4}\pi+\frac{1}{2}\theta)$ in a series of sines of ascending multiples of θ.

4. If $0<\theta<\frac{1}{2}\pi$, prove that

$$\log\cos\theta=-\log 2+\cos 2\theta-\tfrac{1}{2}\cos 4\theta+\tfrac{1}{3}\cos 6\theta-\dots.$$

5. In any triangle, where $a<c$, show that

$$\frac{\cos nA}{b^n}=\frac{1}{c^n}\left\{1+\frac{na}{c}\cos B+\frac{n(n+1)}{1.2}\frac{a^2}{c^2}\cos 2B+\dots\right\}$$

and $\dfrac{\sin nA}{b^n}=\dfrac{n}{c^n}\left\{\dfrac{a}{c}\sin B+\dfrac{n+1}{1.2}\dfrac{a^2}{c^2}\sin 2B+\dots\right\}.$

[*Lucknow*, 1967]

[Solution. In any triangle ABC, we have

$$a\cos B+b\cos A=c, \qquad (1)$$

$$a\sin B-b\sin A=0 \qquad (2)$$

Multiplying (2) by i and adding to (1),

$$a(\cos B+i\sin B)+b(\cos A-i\sin A)=c.$$

or $$ae^{Bi}+be^{-Ai}=c.$$

$\therefore$ $$be^{-Ai}=c\left(1-\frac{a}{c}e^{Bi}\right).$$

Raising both sides to the power $-n$,

$$\frac{e^{Ani}}{b^n}=\frac{1}{c^n}\left(1-\frac{a}{e}e^{Bi}\right)^{-n}$$

$$=\frac{1}{c^n}\left\{1+\frac{na}{c}e^{Bi}+\frac{n(n+1)}{1.2}\frac{a^2}{c^2}e^{2Bi}+\dots\right\},$$

since $a<c$,

or, using Euler's theorem,

$$\frac{1}{b^n}(\cos nA+i\sin nA)=\frac{1}{c^n}\left\{1+\frac{na}{c}(\cos B+i\sin B)\right.$$

$$\left.+\frac{n(n+1)}{1.2}\frac{a^2}{c^2}(\cos 2B+i\sin 2B)+\dots\right\}.$$

Hence, equating real and imaginary parts on each side, we have the desired results.]

6. If $0<x<\sqrt{2}-1$, prove that

$$2(x-\tfrac{1}{3}x^3+\tfrac{1}{5}x^5-\dots)$$
$$=\frac{2x}{1-x^2}-\frac{1}{3}\left(\frac{2x}{1-x^2}\right)^3+\frac{1}{5}\left(\frac{2x}{1-x^2}\right)^5-\dots. \quad [Agra, 1950]$$

7. When both θ and $\tan^{-1}(\sec\theta)$ lie between 0 and $\frac{1}{2}\pi$, prove that

$$\tan^{-1}(\sec\theta)=\tfrac{1}{4}\pi+\tan^2\tfrac{1}{2}\theta-\tfrac{1}{3}\tan^6\tfrac{1}{2}\theta+\tfrac{1}{5}\tan^{10}\tfrac{1}{2}\theta-\dots.$$

8. If $\tan\theta=x+\tan\alpha$, show that

$$\theta=\alpha+x\cos^2\alpha-\tfrac{1}{2}x^2\cos^2\alpha\sin 2\alpha-\tfrac{1}{3}x^3\cos^3\alpha\cos 3\alpha$$
$$+\tfrac{1}{4}x^4\cos^4\alpha\sin 4\alpha-\dots. \quad [Agra, 1941]$$

9. Find the value of π

(i) to 3 decimal places by using Euler's series,

(ii) to 4 decimal places by using Machin's series,

and (iii) to 5 decimal places by using Rutherford's series.

10. In any triangle, if $a-b$ be small compared with c, show that the circular measure of $A-B$ is nearly equal to

$$2\frac{a-b}{c}\sin B+\left(\frac{a-b}{c}\right)^2\sin 2B \text{ nearly.}$$

CHAPTER IX

SUMMATION OF SERIES

9·1. Introduction. In this chapter we shall give some of the important methods of summing up trigonometrical series, finite or infinite.

9·2. Angles in arithmetical progression. Sine series. *To find the sum of a series of sines of angles in arithmetical progression.*

Let the series be

$$\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\ldots+\sin\{\alpha+(n-1)\beta\},$$

consisting of n terms.

Suppose that S_n is the required sum. Then

$$2\sin\tfrac{1}{2}\beta\,.\,S_n=2\sin\tfrac{1}{2}\beta\sin\alpha+2\sin\tfrac{1}{2}\beta\sin(\alpha+\beta)+\ldots+2\sin\tfrac{1}{2}\beta\sin\{\alpha+(n-1)\beta\}.$$

But $2\sin\tfrac{1}{2}\beta\sin\alpha=\cos(\alpha-\tfrac{1}{2}\beta)-\cos(\alpha+\tfrac{1}{2}\beta)$,

$$2\sin\tfrac{1}{2}\beta\sin(\alpha+\beta)=\cos(\alpha+\tfrac{1}{2}\beta)-\cos(\alpha+\tfrac{3}{2}\beta),$$

$$\ldots\qquad\ldots\qquad\ldots$$

$$2\sin\tfrac{1}{2}\beta\sin\{\alpha+(n-2)\beta\}=\cos\{\alpha+(n-\tfrac{5}{2}\beta)\}-\cos\{\alpha+(n-\tfrac{3}{2})\beta\},$$

and

$$2\sin\tfrac{1}{2}\beta\sin\{\alpha+(n-1)\beta\}=\cos\{\alpha+(n-\tfrac{3}{2})\beta\}-\cos\{\alpha+(n-\tfrac{1}{2})\beta\}.$$

By addition,

$$2\sin\tfrac{1}{2}\beta\,.\,S_n=\cos(\alpha-\tfrac{1}{2}\beta)-\cos\{\alpha+(n-\tfrac{1}{2})\beta\}$$
$$=2\sin\{\alpha+\tfrac{1}{2}(n-1)\beta\}\sin\tfrac{1}{2}n\beta.$$

Therefore $S_n = \dfrac{\sin\{\alpha+\frac{1}{2}(n-1)\beta\}\sin\frac{1}{2}n\beta}{\sin\frac{1}{2}\beta}$.

As a particular case, when $\beta=\alpha$, we have

$$\sin\alpha+\sin 2\alpha+\sin 3\alpha+\ldots+\sin n\alpha$$
$$=\frac{\sin\frac{1}{2}(n+1)\alpha\sin\frac{1}{2}n\alpha}{\sin\frac{1}{2}\alpha}.$$

9·21. A particular case. If we take $\beta=2\pi/n$, the sum of the series in §9·2 is zero, since $\sin\frac{1}{2}n\beta=\sin\pi=0$. In other words,

$$\sin\alpha+\sin(\alpha+2\pi/n)+\sin(\alpha+4\pi/n)+\ldots$$
$$+\sin\{\alpha+2(n-1)\pi/n\}$$

is zero, whatever be the value of $n(>1)$.

9·3. Angles in arithmetical progression. Cosine series. *To find the sum of a series of cosines of angles in arithmetical progression.*

Let the series be

$$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\ldots$$
$$+\cos\{\alpha+(n-1)\beta\}.$$

If the sum be S_n, then

$$2\sin\tfrac{1}{2}\beta\,.\,S_n=2\sin\tfrac{1}{2}\beta\cos\alpha+2\sin\tfrac{1}{2}\beta\cos(\alpha+\beta)$$
$$+\ldots+2\sin\tfrac{1}{2}\beta\cos\{\alpha+(n-1)\beta\}$$
$$=\sin(\alpha+\tfrac{1}{2}\beta)-\sin(\alpha-\tfrac{1}{2}\beta)$$
$$+\sin(\alpha+\tfrac{3}{2}\beta)-\sin(\alpha+\tfrac{1}{2}\beta)$$
$$\ldots\qquad\ldots\qquad\ldots$$
$$+\sin\{\alpha+(n-\tfrac{3}{2})\beta\}-\sin\{\alpha+(n-\tfrac{5}{2})\beta\}$$
$$+\sin\{\alpha+(n-\tfrac{1}{2})\beta\}-\sin\{\alpha+(n-\tfrac{3}{2})\beta\},$$

by virtue of the identity

$$2\sin\tfrac{1}{2}\beta\cos(\alpha+r\beta)=\sin\{\alpha+(r+\tfrac{1}{2})\beta\}$$
$$-\sin\{\alpha+(r-\tfrac{1}{2})\beta\}.$$

Adding the terms on the right hand side, we have

$$2 \sin \tfrac{1}{2}\beta \,.\, S_n = \sin \{\alpha + (n-\tfrac{1}{2})\beta\} - \sin (\alpha - \tfrac{1}{2}\beta\}$$
$$= 2 \cos \{\alpha + \tfrac{1}{2}(n-1)\beta\} \sin \tfrac{1}{2}n\beta.$$

Therefore

$$S_n = \frac{\cos \{\alpha + \frac{1}{2}(n-1)\beta\} \sin \frac{1}{2}n\beta}{\sin \frac{1}{2}\beta}. \qquad (1)$$

As a particular case, taking $\beta = \alpha$, we have

$$\cos \alpha + \cos 2\alpha + \cos 3\alpha + \ldots + \cos n\alpha$$
$$= \frac{\cos \frac{1}{2}(n+1)\alpha \sin \frac{1}{2}n\alpha}{\sin \frac{1}{2}\alpha}.$$

9·31. A particular case of the cosine series. For reasons given in §9·21, we have from (1) of § 9·3,

$$\cos \alpha + \cos (\alpha + 2\pi/n) + \cos (\alpha + 4\pi/n) + \ldots$$
$$+ \cos \{\alpha + 2(n-1)\pi/n\} = 0.$$

9·32. Aid to memory. The sums of the sine and cosine series of § 9·2 and § 9·3 may be easily committed to memory if remembered in the following form :

Sum of the sine-series

$$= \frac{\sin \left\{\dfrac{\mathbf{n \times diff}}{2}\right\}}{\sin \left\{\dfrac{\mathbf{diff}}{2}\right\}} \sin \left\{\frac{\textbf{first angle} + \textbf{last angle}}{2}\right\}.$$

Sum of the cosine-series

$$= \frac{\sin \left\{\dfrac{\mathbf{n \times diff}}{2}\right\}}{\sin \left\{\dfrac{\mathbf{diff}}{2}\right\}} \cos \left\{\frac{\textbf{first angle} + \textbf{last angle}}{2}\right\}.$$

9·4. When the common difference is a sub-multiple of 2π. Since any power of the sine or cosine of an angle can be expressed in a series of sines or cosines of multiples of the angle (§7·21, § 7·22), it follows from § 9·21 and § 9·31 that, when $\beta = 2\pi/n$, the sum of the series

$$\sin^m \alpha + \sin^m (\alpha+\beta) + \sin^m (\alpha+2\beta) + \ldots + \sin^m \{\alpha+(n-1)\beta\}$$

or

$$\cos^m \alpha + \cos^m (\alpha+\beta) + \cos^m (\alpha+2\beta) + \ldots + \cos^m \{\alpha+(n-1)\beta\},$$

when $m<n$, is independent of the angles.

For example, taking $m=4$, we have

$$\cos^4 \alpha = 2^{-3}[\cos 4\alpha + 4\cos 2\alpha + 3],$$

$$\cos^4 (\alpha+\beta) = 2^{-3}[\cos (4\alpha+4\beta) + 4\cos (2\alpha+2\beta) + 3],$$

and so on.

Hence $\cos^4 \alpha + \cos^4 (\alpha+\beta) + \cos^4 (\alpha+2\beta) + \ldots + \cos^4 \{\alpha+(n-1)\beta\}$

$$= 2^{-3}\,[\cos 4\alpha + \cos (4\alpha+4\beta\} + \cos (4\alpha+8\beta) + \ldots + \cos \{4\alpha+(n-1)4\beta\}]$$

$$+\tfrac{1}{2}[\cos 2\alpha + \cos (2\alpha+2\beta) + \cos (2\alpha+4\beta) + \ldots + \cos \{2\alpha+(n-1)2\beta\}] + \tfrac{3}{8}n.$$

But if $\beta = 2\pi/n$, and $n>4$, the sum of each of the two series within brackets is zero, and we get

$$\cos^4 \alpha + \cos^4 (\alpha+2\pi/n) + \cos^4 (\alpha+4\pi/n) + \ldots + \cos^4 \{\alpha+2(n-1)\pi/n\}$$

$$= \tfrac{3}{8}n.$$

The condition that m is less than n is necessary to avoid the denominators in the expressions for the sums of the sines and cosines becoming zero.

Ex. Find the sum to n terms of the series

$\sin\alpha\sin 2\alpha+\sin 2\alpha\sin 3\alpha+\sin 3\alpha\sin 4\alpha+\ldots,$

and deduce the sum of

$$1\,.\,2+2\,.\,3+\ldots+n(n+1). \qquad [\textit{Lucknow}, 1957]$$

We have

$$\sin\alpha\sin 2\alpha=\tfrac{1}{2}[\cos\alpha-\cos 3\alpha]$$
$$\sin 2\alpha\sin 3\alpha=\tfrac{1}{2}[\cos\alpha-\cos 5\alpha]$$
$$\ldots \qquad \ldots \qquad \ldots$$
$$\sin n\alpha\sin(n+1)\alpha=\tfrac{1}{2}[\cos\alpha-\cos(2n+1)\alpha].$$

Hence, on adding, we have

$$\sin\alpha\sin 2\alpha+\sin 2\alpha\sin 3\alpha+\ldots+\sin n\alpha\sin(n+1)\alpha$$
$$=\tfrac{1}{2}n\cos\alpha-\tfrac{1}{2}\{\cos 3\alpha+\cos 5\alpha+\ldots+\cos(2n+1)\alpha\}$$
$$=\tfrac{1}{2}n\cos\alpha-\tfrac{1}{2}\cos\tfrac{1}{2}\{3\alpha+(2n+1)\alpha\}\sin n\alpha\operatorname{cosec}\alpha,$$

where we have used (1) of § 9·3,

$$=\tfrac{1}{2}n\cos\alpha-\tfrac{1}{2}\sin n\alpha\cos(n+2)\alpha\operatorname{cosec}\alpha$$
$$=\tfrac{1}{2}n\cos\alpha-\tfrac{1}{4}[\sin(2n+2)\alpha-\sin 2\alpha]\operatorname{cosec}\alpha.$$
$$=[\tfrac{1}{4}(n+1)\sin 2\alpha-\sin(2n+2)\alpha]\operatorname{cosec}\alpha. \qquad (1)$$

To find the sum of

$$1\,.\,2+2\,.\,3+\ldots+n(n+1),$$

we notice that $1\,.\,2$ is the coefficient of α^2 in the expansion of $\sin\alpha\sin 2\alpha$ in powers of α, $2\,.\,3$ is the coefficient of α^2 in that of $\sin 2\alpha\sin 3\alpha$, and so on. Hence the required sum is the coefficient of α^2 in (1), i.e., the coefficient of α^3 in

$$\tfrac{1}{4}[(n+1)\sin 2\alpha-\sin(2n+2)\alpha],$$

which is

$$\tfrac{1}{4}\left[(n+1)\left(-\frac{8}{3!}\right)-\left(-\frac{(2n+2)^3}{3!}\right)\right\}$$
$$=(2/3!)[(n+1)^3-(n+1)]$$
$$=\tfrac{1}{3}n(n+1)(n+2).$$

EXAMPLES

1. Find the sum to n terms of the series

(i) $\sin\alpha-\sin(\alpha+\beta)+\sin(\alpha+2\beta)-\sin(\alpha+3\beta)+\ldots$.

[*Roorkee*, '62]

(ii) $\cos\theta+\cos 3\theta+\cos 5\theta+\dots$.
(iii) $\sin^2 \alpha+\sin^2(\alpha+\beta)+\sin^2(\alpha+2\beta)+\dots$.
(iv) $\cos^2\theta+\cos^2 2\theta+\cos^2 3\theta+\dots$.
(v) $\sin^3 \alpha+\sin^3 2\alpha+\sin^3 3\alpha+\dots$.
(vi) $\sin^4 \alpha+\sin^4 2\alpha+\sin^4 3\alpha+\dots$.
(vii) $\cos\theta\cos 3\theta+\cos 3\theta\cos 5\theta+\cos 5\theta\cos 7\theta+\dots$.

2. Prove that

(i) $$\frac{\sin\alpha+\sin 2\alpha+\dots+\sin n\alpha}{\cos\alpha+\cos 2\alpha+\dots+\cos n\alpha}=\tan\tfrac{1}{2}(n+1)\alpha.$$

(ii) $$\frac{\sin\alpha-\sin(\alpha+\beta)+\sin(\alpha+2\beta)-\dots\text{to } n \text{ terms})}{\cos\alpha-\cos(\alpha+\beta)+\cos(\alpha+2\beta)-\dots\text{to } n \text{ terms}}.$$
$$=\tan\{\alpha+\tfrac{1}{2}(n-1)(\pi+\beta)\}.$$

3. Find the sum of the series
$$\sin\alpha+\sin 2\alpha+\sin 3\alpha+\dots+\sin n\alpha,$$
and hence deduce the sum of the series
$$1+2+3+\dots+n.$$ [*Calcutta*, '51; *Banaras*, '63]

4. Prove that,
$$\cos\tfrac{1}{11}\pi+\cos\tfrac{3}{11}\pi+\cos\tfrac{5}{11}\pi+\cos\tfrac{7}{11}\pi+\cos\tfrac{9}{11}\pi=\tfrac{1}{2}.$$

5. If n is an integer greater than 2, prove that
$$\sin(2\pi/n)+\sin(4\pi/n)+\sin(6\pi/n)+\dots+\sin(2n\pi/n)=0.$$

6. Sum the following series :
$$\cos^2\alpha+\cos^2(\alpha+\pi/2)+\cos^2(\alpha+2\pi/2)+\dots\text{to } n \text{ terms.}$$ [*Luck.*, '58]

7. Find the sum to n terms of the series
$$\cos^3\alpha+\cos^3(\alpha+\beta)+\cos^3(\alpha+2\beta)+\dots,$$
when $\beta=2\pi/n$ and $n>3$. [*Lucknow*, 1968]

9·5. A general method. A general method of summation of trigonometrical series, which covers many of the cases, is what may be called the $C+iS$ method.

Suppose that it is required to find the sum of one of the series, finite or infinite,
$$C=a_0\cos\alpha+a_1\cos(\alpha+\beta)+a_2\cos(\alpha+2\beta)+\dots$$
and $$S=a_0\sin\alpha+a_1\sin(\alpha+\beta)+a_2\sin(\alpha+2\beta)+\dots.$$

If we want to find the sum of the series of cosines, the series of sines will be called the auxiliary series. In case the sum of the series of sines is required, the series of cosines will be called the auxiliary series.

We multiply the series of sines by i and add to the series of cosines. We then get

$$\begin{aligned} C+iS &= a_0(\cos\alpha+i\sin\alpha) \\ &\quad +a_1\{\cos(\alpha+\beta)+i\sin(\alpha+\beta)\} \\ &\quad +a_2\{\cos(\alpha+2\beta)+i\sin(\alpha+2\beta)\}+\ldots \\ &= a_0e^{i\alpha}+a_1e^{i(\alpha+\beta)}+a_2e^{i(\alpha+2\beta)}+\ldots. \end{aligned}$$

The last series can, in general, be summed up, if it be in one of the following forms :

(i) Series in geometrical progression.

(ii) Binomial series or one which can be reduced to it.

(iii) Exponential series or its sub-case sine or cosine series.

(iv) Logarithmic series or its sub-case Gregory's series.

The sum thus obtained is expressed in the form $A+iB$ and equating the real and imaginary parts, we get that C, the series of cosines, is equal to A and S, the series of sines, is equal to B.

Ex. 1. Find the sum of the series

$$\sin\alpha+x\sin(\alpha+\beta)+(x^2/2\,!)\sin(\alpha+2\beta)+\ldots$$

[*Lucknow*, 1967]

Let $S=\sin\alpha+x\sin(\alpha+\beta)+(x^2/2\,!)\sin(\alpha+2\beta)+\ldots,$ and suppose that

$$C=\cos\alpha+x\cos(\alpha+\beta)+(x^2/2\,!)\cos(\alpha+2\beta)+\ldots.$$

Then

$$C+iS=(\cos\alpha+i\sin\alpha)+x\{\cos(\alpha+\beta)+i\sin(\alpha+\beta)\}$$
$$+(x^2/2!)\{\cos(\alpha+2\beta)+i\sin(\alpha+2\beta)\}+\dots.$$
$$=e^{i\alpha}+xe^{i(\alpha+\beta)}+(x^2/2!)e^{i(\alpha+2\beta)}+\dots$$
$$=e^{i\alpha}[1+xe^{i\beta}+(xe^{i\beta})^2/2!+\dots]. \quad (1)$$

The series within the square brackets is an exponential series and hence

$$C+iS=e^{i\alpha}\,.\,e^{xe^{i\beta}}$$
$$=e^{i\alpha}\,.\,e^{x(\cos\beta+i\sin\beta)}$$
$$=e^{x\cos\beta}\,.\,e^{i(\alpha+x\sin\beta)}$$
$$=e^{x\cos\beta}[\cos(\alpha+x\sin\beta)+i\sin(\alpha+x\sin\beta)].$$

Equating the imaginary parts on the two sides, we have

$$S=e^{x\cos\beta}\sin(\alpha+x\sin\beta).$$

The series within the square brackets in (1) is convergent when x is numerically less than unity and the sum is true under this condition.

Ex. 2. Find the sum to infinity of the series

$$1+\tfrac{1}{2}\cos 2\theta-\frac{1}{2.4}\cos 4\theta+\frac{1.3}{2.4.6}\cos 6\theta$$
$$-\frac{1.3.5}{2.4.6.8}\cos 8\theta+\dots. \qquad \text{[\textit{Lucknow}, 1968]}$$

Let

$$C=1+\tfrac{1}{2}\cos 2\theta-\frac{1}{2.4}\cos 4\theta+\frac{1.3}{2.4.6}\cos 6\theta-\dots$$

and suppose that

$$S=\tfrac{1}{2}\sin 2\theta-\frac{1}{2.4}\sin 4\theta+\frac{1.3}{2.4.6}\sin 6\theta-\dots.$$

Then

$$C+iS=1+\tfrac{1}{2}(\cos 2\theta+i\sin 2\theta)-\frac{1}{2.4}(\cos 4\theta+i\sin 4\theta)$$
$$+\frac{1.3}{2.4.6}(\cos 6\theta+i\sin 6\theta)-\dots$$
$$=1+\tfrac{1}{2}e^{2i\theta}-\frac{1}{2.4}e^{4i\theta}+\frac{1.3}{2.4.6}e^{6i\theta}-\dots.$$

This is a binomial series and we have

$$\begin{aligned} C+iS &= (1+e^{2i\theta})^{1/2} \\ &= (1+\cos 2\theta+i\sin 2\theta)^{1/2} \\ &= (2\cos^2\theta+2i\sin\theta\cos\theta)^{1/2} \\ &= \sqrt{(2\cos\theta)}\,(\cos\theta+i\sin\theta)^{1/2}. \end{aligned}$$

Taking the principal value, we have

$$C+iS=\sqrt{(2\cos\theta)}(\cos\tfrac{1}{2}\theta+i\sin\tfrac{1}{2}\theta).$$

Equating the real parts on the two sides, we get

$$C=\sqrt{(2\cos\theta)}\cos\tfrac{1}{2}\theta.$$

Ex. 3. Sum the series

$1+x\cos\theta+x^2\cos 2\theta+\ldots+x^{n-1}\cos(n-1)\theta.$

[*Utkal*, 1953; *Agra*, 1952]

Let $C=1+x\cos\theta+x^2\cos 2\theta+\ldots+x^{n-1}\cos(n-1)\theta$, and suppose that

$$S=x\sin\theta+x^2\sin 2\theta+\ldots+x^{n-1}\sin(n-1)\theta.$$

Then $C+iS=1+xe^{i\theta}+x^2e^{2i\theta}+\ldots+x^{n-1}\,e^{i(n-1)\theta}$.

This is a series in geometrical progression, so that

$$C+iS=\frac{1-x^ne^{in\theta}}{1-xe^{i\theta}}.$$

To separate the real and imaginary parts of this sum, we multiply the numerator and the denominator by $1-xe^{-i\theta}$. We then have

$$\begin{aligned} C+iS &= \frac{(1-x^ne^{in\theta})(1-xe^{-i\theta})}{(1-xe^{i\theta})(1-xe^{-i\theta})} \\ &= \frac{1-xe^{-i\theta}-x^n e^{in\theta}+x^{n+1}e^{i(n-1)\theta}}{1-2x\cos\theta+x^2}. \end{aligned}$$

On equating the real parts on either side, we have

$$C=\frac{1-x\cos\theta-x^n\cos n\theta+x^{n+1}\cos(n-1)\theta}{1-2x\cos\theta+x^2}.$$

Ex. 4. If $C=\cos^2\theta-\frac{1}{3}\cos^3\theta\cos 3\theta+\frac{1}{5}\cos^5\theta\cos 5\theta-\ldots$, show that $\tan 2C=2\cot^2\theta.$ [*Travancore*, 1948]

Suppose that

$$S = \cos\theta \sin\theta - \tfrac{1}{3}\cos^3\theta \sin 3\theta + \tfrac{1}{5}\cos^5\theta \sin 5\theta - \ldots,$$

then
$$C + iS = \cos\theta\, e^{i\theta} - \tfrac{1}{3}\cos^3\theta\, e^{3\theta i} + \tfrac{1}{5}\cos^5\theta\, e^{5i\theta} - \ldots$$
$$= \tan^{-1}(e^{i\theta}\cos\theta),$$

by Gregory's series.

Hence $\qquad \tan(C+iS) = \cos\theta \,.\, e^{i\theta}$

and accordingly $\qquad \tan(C-iS) = \cos\theta \,.\, e^{-i\theta}.$

Therefore $\quad \tan 2C = \tan\{(C+iS)+(C-iS)\}$

$$= \frac{\tan(C+iS)+\tan(C-iS)}{1-\tan(C+iS)\tan(C-iS)}$$

$$= \frac{\cos\theta\, e^{i\theta} + \cos\theta\, e^{-i\theta}}{1-\cos^2\theta}$$

$$= \frac{2\cos^2\theta}{\sin^2\theta}$$

$$= 2\cot^2\theta.$$

9·6. Use of the exponential values. Sometimes we may write the exponential value of the sine or cosine occurring in the series and then sum up the two series obtained.

Ex. Sum the series

$$c\sin a - \tfrac{1}{3}c^3 \sin 3a + \tfrac{1}{5}c^5 \sin 5a - \ldots . \qquad [\textit{Agra}, 1937]$$

The given series, convergent when c is numerically less than unity,

$$= \frac{c}{2i}(e^{i\alpha} - e^{-i\alpha}) - \frac{c^3}{3\,.\,2i}(e^{3i\alpha} - e^{-3i\alpha})$$
$$+ \frac{c^5}{5\,.\,2i}(e^{5i\alpha} - e^{-5i\alpha}) - \ldots$$

$$= -\tfrac{1}{2}i[(ce^{i\alpha} - \tfrac{1}{3}c^3 e^{3i\alpha} + \tfrac{1}{5}c^5 e^{5i\alpha} - \ldots)$$
$$-(ce^{-i\alpha} - \tfrac{1}{3}c^3 e^{-3i\alpha} + \tfrac{1}{5}c^5 e^{-5i\alpha} - \ldots)].$$

Each of the series is a Gregory's series and we have the required sum

$$= -\tfrac{1}{2}i[\tan^{-1}(ce^{i\alpha}) - \tan^{-1}(ce^{-i\alpha})].$$

If $\tan^{-1}(ce^{i\alpha}) = A + iB$,

then $\tan(A+iB) = ce^{i\alpha}$;

and hence $\tan(A-iB) = ce^{-i\alpha}$.

Therefore
$$\tan 2iB = \tan\{(A+iB)-(A-iB)\}$$
$$= \frac{\tan(A+iB) - \tan(A-iB)}{1+\tan(A+iB)\tan(A-iB)}$$
$$= \frac{ce^{i\alpha} - ce^{-i\alpha}}{1+c^2} = \frac{2ic\sin\alpha}{1+c^2}.$$

Or
$$-i\frac{e^{-2B}-e^{2B}}{e^{2B}+e^{-2B}} = \frac{2ic\sin\alpha}{1+c^2},$$

or
$$\frac{e^{2B}-e^{-2B}}{e^{2B}+e^{-2B}} = \frac{2c\sin\alpha}{1+c^2}.$$

Therefore
$$e^{4B} = \frac{1+2c\sin\alpha+c^2}{1-2c\sin\alpha+c^2},$$

giving
$$B = \tfrac{1}{4}\log\frac{1+2c\sin\alpha+c^2}{1-2c\sin\alpha+c^2}.$$

Hence the required sum
$$= -\tfrac{1}{2}i[(A+iB)-(A-iB)]$$
$$= B$$
$$= \tfrac{1}{4}\log\frac{1+2c\sin\alpha+c^2}{1-2c\sin\alpha+c^2}.$$

Examples

Sum the series

1. $\cos\theta - \frac{1}{2}\cos 2\theta + \frac{1}{3}\cos 3\theta - \ldots$ ad inf. [*Banaras*, '52; *Lucknow*, '66]

2. $\sin\alpha + \frac{1}{2!}\sin 2\alpha + \frac{1}{3!}\sin 3\alpha + \ldots$ ad inf. [*Lucknow*, '45]

3. $\sin\alpha + \frac{1}{2}\sin 2\alpha + \frac{1}{2^2}\sin 3\alpha + \ldots$ ad inf. [*Lucknow*, '66]

4. $x\sin\theta - \frac{1}{2}x^2\sin 2\theta + \frac{1}{3}x^3\sin 3\theta - \ldots$ ad inf., given that $|x| < 1$. [*Banaras*, '62]

5. $x\cos\alpha + \frac{1}{3}x^3\cos 3\alpha + \frac{1}{5}x^5\cos 5\alpha + \ldots$ ad inf. [*Travancore*, '50]

$-\frac{1}{2}(-\frac{1}{2}-1)$

$\frac{1}{2}\cdot\frac{3}{2}$

6. $1+x\cos\theta+x^2\cos 2\theta+x^3\cos 3\theta+\ldots$ ad inf., given that $|x|<1$. [*Annam.*, '51]

7. $1-\frac{1}{2}\cos\theta+\frac{1\cdot 3}{2.4}\cos 2\theta-\frac{1.3.5}{2.4.6}\cos 3\theta+\ldots$, $(-\pi<\theta<\pi)$. [*Luck.*, '59]

8. $\cos\theta-\frac{1}{3}\cos 3\theta+\frac{1}{5}\cos 5\theta-\ldots$. [*Alld.*, '50]

9. $\sin\theta-\frac{1}{3!}\sin 3\theta+\frac{1}{5!}\sin 5\theta-\ldots$ ad inf.

10. $\sin\alpha+\frac{1}{2}\sin 3\alpha+\frac{1.3}{2.4}\sin 5\alpha+\ldots$ ad inf. [*Sagar*, '51; *Luck.*, '55]

11. $\frac{1}{2}\sin\alpha+\frac{1\cdot 3}{2.4}\sin 2\alpha+\frac{1.3.5}{2.4.6}\sin 3\alpha+\ldots$ ad inf. [*Lucknow*, '60; *Banaras*, '61]

Cosh $(xe^{i\theta})$

Cos $(xie^{i\theta})$

12. $1+\frac{x^2\cos 2\theta}{2!}+\frac{x^4\cos 4\theta}{4!}+\ldots$ ad inf. [*Luck.*, 67]

13. $x\sin\alpha+\frac{1}{2}x^2\sin 2\alpha+\frac{1}{3}x^3\sin 3\alpha+\ldots$ ad inf., given that $|x|<1$. [*Agra*, '53]

14. $1-\cos\alpha\cos\beta+\frac{\cos^2\alpha}{2!}\cos 2\beta-\frac{\cos^3\alpha}{3!}\cos 3\beta+\ldots$ ad inf. [*Delhi*, '53]

$|1-\cos e^{i\theta}|$ $=\sin^{-1}[$

15. $1+\cos\theta\cos\theta+\cos^2\theta\cos 2\theta+\cos^3\theta\cos 3\theta+\ldots$ ad inf.

16. $\sin\theta\cos\theta+\frac{\sin 2\theta\cos^2\theta}{2!}+\frac{\sin 3\theta\cos^3\theta}{3!}+\ldots$ [*Luck.*, '56]

17. $\cos\theta+\frac{\sin\theta}{1!}\cos 2\theta+\frac{\sin^2\theta}{2!}\cos 3\theta+\ldots$. [*Luck.*, '64]

18. $2x\cos\theta+\frac{3}{2}x^2\cos^2\theta+\frac{4}{3}x^3\cos^3\theta+\frac{5}{4}x^4\cos^4\theta+\ldots$ ad inf., given that $|x|<1$. [*Lucknow*. '59]

19. $\cos(\pi/3)+\frac{1}{3}\cos(2\pi/3)+\frac{1}{5}\cos(3\pi/3)+\frac{1}{7}\cos(4\pi/3)+\ldots$ ad inf. [*Luck.*, '62]

[Let $C=\cos(\pi/3)+\frac{1}{3}\cos(2\pi/3)+\frac{1}{5}\cos(3\pi/3)+\ldots$,

and $S=\sin(\pi/3)+\frac{1}{3}\sin(2\pi/3)+\frac{1}{5}\sin(3\pi/3)+\ldots$

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Then

$$C+iS=e^{\pi i/3}+\tfrac{1}{3}e^{2\pi i/3}+\tfrac{1}{5}e^{3\pi i/3}+\dots$$
$$=e^{2\pi i/6}+\tfrac{1}{3}e^{4\pi i/6}+\tfrac{1}{5}e^{6\pi i/6}+\dots$$
$$=e^{\pi i/6}[e^{\pi i/6}+\tfrac{1}{3}e^{3\pi i/6}+\tfrac{1}{5}e^{5\pi i/6}+\dots)$$
$$=e^{\pi i/6}\cdot\tfrac{1}{2}[\log(1+e^{\pi i/6})-\log(1-e^{\pi i/6})].$$

Now

$$e^{\pi i/6}=\cos(\pi/6)+i\sin(\pi/6)=\sqrt{3}/2+i/2,$$

and

$$\log(1+e^{\pi i/6})-\log(1-e^{\pi i/6})=\log\frac{2+\sqrt{3}+i}{2-\sqrt{3}-i}$$
$$=\log\sqrt{\{(2+\sqrt{3})^2+1\}}+i\tan^{-1}\frac{1}{2+\sqrt{3}}$$
$$-\log\sqrt{\{(2-\sqrt{3})^2+1\}}+i\tan^{-1}\frac{1}{2-\sqrt{3}}$$
$$=\tfrac{1}{2}\log\frac{8+4\sqrt{3}}{8-4\sqrt{3}}+i\frac{\pi}{2}$$
$$=\tfrac{1}{2}\log\frac{2+\sqrt{3}}{2-\sqrt{3}}+i\frac{\pi}{2}$$
$$=\log(2+\sqrt{3})+i\pi/2.$$

Therefore

$$C+iS=\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)\cdot\tfrac{1}{2}\left[\log(2+\sqrt{3})+\frac{i\pi}{2}\right].$$

Hence, equating real parts on the two sides,

$$C=\frac{\sqrt{3}}{4}\log(2+\sqrt{3})-\pi/8$$
$$=\tfrac{1}{8}\{2\sqrt{3}\log(2+\sqrt{3})-\pi\}.$$

20. Prove that

$$\sum_{n=1}^{m}\cos^n\theta\cos n\theta=\frac{\cos^{m+1}\theta\sin m\theta}{\sin\theta}.$$ [*Lucknow*, '54]

9·7. Series of hyperbolic functions. A series of hyperbolic sines or cosines can be summed

up either directly or by using their values in terms of exponential functions.

Ex. Sum the series

$$\cosh\theta+\frac{\sin\theta}{1!}\cosh 2\theta+\frac{\sin^2\theta}{2!}\cosh 3\theta+\ldots.$$

[*Nagpur*, '40]

The given series

$$=\tfrac{1}{2}(e^{\theta}+e^{-\theta})+(1/1!)\sin\theta\,.\,\tfrac{1}{2}(e^{2\theta}+e^{-2\theta})$$
$$+(1/2!)\sin^2\theta\,.\,\tfrac{1}{2}(e^{3\theta}+e^{-3\theta})+\ldots$$
$$=\tfrac{1}{2}[\{e^{\theta}+(1/1!)\sin\theta\, e^{2\theta}+(1/2!)\sin^2\theta\, e^{3\theta}+\ldots\}$$
$$+\{e^{-\theta}+(1/1!)\sin\theta\, e^{-2\theta}+(1/2!)\sin^2\theta\, e^{-3\theta}+\ldots\}]$$
$$=\tfrac{1}{2}(e^{\theta}\{1+(1/1!)e^{\theta}\sin\theta+(1/2!)e^{2\theta}\sin^2\theta+\ldots\}$$
$$+e^{-\theta}\{1+(1/1!)e^{-\theta}\sin\theta+(1/2!)e^{-2\theta}\sin^2\theta+\ldots\})$$
$$=\tfrac{1}{2}[e^{\theta}\,e^{e^{\theta}\sin\theta}+e^{-\theta}\,e^{e^{-\theta}\sin\theta}]$$
$$=\tfrac{1}{2}[e^{\theta+e^{\theta}\sin\theta}+e^{-\theta+e^{-\theta}\sin\theta}].$$

EXAMPLES

Sum the following series :

1. $\sinh\theta+\sinh 2\theta+\sinh 3\theta+\ldots$to n terms.
2. $\cosh x+\cosh(x+y)+\cosh(x+2y)+\ldots$to n terms.
3. $\cosh^2\alpha+\cosh^2(\alpha+\beta)+\cosh^2(\alpha+2\beta)+\ldots$to n terms. [*Lucknow*, '64]
4. $1+\cosh x+(1/2!)\cosh 2x+(1/3!)\cosh 3x+\ldots$ad inf. [*Lucknow*, '66]
5. $x\cosh\theta+\dfrac{x^2}{2!}\cosh 2\theta+\dfrac{x^3}{3!}\cosh 3\theta+\ldots$ad inf.

9·8. The difference method. This method consists in splitting each term as a difference of two expressions in a suitable manner. Thus if we want the sum of

$$u_1+u_2+u_3+\ldots+u_r+\ldots+u_n,$$

we write $$u_r=f(r+1)-f(r).$$

Then
$$u_1 = f(2) - f(1),$$
$$u_2 = f(3) - f(2),$$
$$u_3 = f(4) - f(3),$$
$$\ldots \quad \ldots \quad \ldots$$
$$u_{n-1} = f(n) - f(n-1),$$
$$u_n = f(n+1) - f(n).$$

On adding, we have
$$u_1 + u_2 + \ldots + u_n = f(n+1) - f(1),$$
since all other terms cancel.

This method can be used to find the sum of an infinite series when the latter is convergent. Thus if the sum of n terms be given by S_n, the sum to infinity will be
$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\{f(n+1) - f(1)\}.$$

Ex. 1. Sum to n terms the series
$$\tan x + \tfrac{1}{2}\tan\frac{x}{2} + \frac{x}{2^2}\tan\frac{x}{2^2} + \ldots.$$
Deduce its value when n increases indefinitely. [*Alld.*, '54]

Since
$$\cot x - \tan x = \frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}$$
$$= \frac{\cos^2 x - \sin^2 x}{\sin x \cos x} = \frac{2\cos 2x}{\sin 2x} = 2\cot 2x,$$
we have
$$\tan x = \cot x - 2\cot 2x.$$

Similarly
$$\tfrac{1}{2}\tan\tfrac{1}{2}x = \tfrac{1}{2}\cot\tfrac{1}{2}x - \cot x,$$
and the general term u_r is given by
$$u_r \equiv \frac{1}{2^{r-2}}\tan\frac{x}{2^{r-1}} = \frac{1}{2^{r-1}}\cot\frac{x}{2^{r-1}} - \frac{1}{2^{r-2}}\cot\frac{x}{2^{r-2}}.$$

Hence
$$S_n = \sum_{r=1}^{n} u_r = \sum_{r=1}^{n} \frac{1}{2^{r-1}}\tan\frac{x}{2^{r-1}}$$

$$=\sum_{r=1}^{n}\left[\frac{1}{2^{r-1}}\cot\frac{1}{2^{r-1}}-\frac{1}{2^{r-2}}\cot\frac{1}{2^{r-2}}\right]$$

$$=\frac{1}{2^{n-1}}\cot\frac{x}{2^{n-1}}-2\cot 2x.$$

To find the sum of an infinite number of terms, we have to evaluate the lim S_n when n tends to infinity. This involves finding the limit of $(1/2^{n-1})\cot(x/2^{n-1})$. For this, put $\theta=x/2^{n-1}$, then

$$\lim_{n\to\infty}\frac{1}{2^{n-1}}\cot\frac{x}{2^{n-1}}=\lim_{\theta\to 0}\frac{\theta}{x}\cot\theta$$

$$=\frac{1}{x}\lim_{\theta\to 0}\cos\theta\cdot\frac{\theta}{\sin\theta}=\frac{1}{x},$$

since $(\sin\theta)/\theta$ and $\cos\theta$ both tend to unity when θ tends to zero.

Hence $$\tan x+\tfrac{1}{2}\tan\frac{x}{2}+\frac{1}{2^2}\tan\frac{x}{2^2}+\ldots\text{ad inf.}$$
$$=1/x-2\cot 2x.$$

Ex. 2. Sum the series

$$\sin^{-1}\frac{1}{\sqrt{2}}+\sin^{-1}\frac{\sqrt{2}-1}{\sqrt{6}}+\sin^{-1}\frac{\sqrt{3}-\sqrt{2}}{\sqrt{12}}+\ldots$$
$$+\sin^{-1}\frac{\sqrt{n}-\sqrt{(n-1)}}{\sqrt{\{n(n+1)\}}}+\ldots\text{ad inf.}$$

[*Lucknow*, '58]

We have $\sin^{-1}\dfrac{\sqrt{n}-\sqrt{(n-1)}}{\sqrt{\{n(n+1)\}}}$

$$=\sin^{-1}\left[\frac{1}{\sqrt{(n+1)}}-\frac{\sqrt{(n-1)}}{\sqrt{n}}\cdot\frac{1}{\sqrt{(n+1)}}\right]$$

$$=\sin^{-1}\left[\frac{1}{\sqrt{n}}\cdot\sqrt{\left(1-\frac{1}{n+1}\right)}-\sqrt{\left(1-\frac{1}{n}\right)}\cdot\frac{1}{\sqrt{(n+1)}}\right]\}$$

$$=\sin^{-1}\frac{1}{\sqrt{n}}-\sin^{-1}\frac{1}{\sqrt{(n+1)}}.$$

Therefore, putting $n=1, 2, 3, \ldots$ successively, we have

$$\sin^{-1}\frac{1}{\sqrt{2}}=\sin^{-1} 1-\sin^{-1}\frac{1}{\sqrt{2}},$$

$$\sin^{-1}\frac{\sqrt{2}-1}{\sqrt{6}}=\sin^{-1}\frac{1}{\sqrt{2}}-\sin^{-1}\frac{1}{\sqrt{3}},$$

$$\ldots \qquad \ldots \qquad \ldots$$

and the nth term $=\sin^{-1}\dfrac{1}{\sqrt{n}}-\sin^{-1}\dfrac{1}{\sqrt{(n+1)}}$.

By adding, the sum to n terms

$$=\sin^{-1} 1-\sin^{-1}\frac{1}{\sqrt{(n+1)}}.$$

Hence the sum to infinity

$$=\lim_{n\to\infty}\left[\sin^{-1} 1-\sin^{-1}\frac{1}{\sqrt{(n+1)}}\right]$$

$$=\tfrac{1}{2}\pi-\sin^{-1} 0=\tfrac{1}{2}\pi.$$

EXAMPLES

Sum the series

1. $\operatorname{cosec} x+\operatorname{cosec} 2x+\operatorname{cosec} 4x+\ldots+\operatorname{cosec} 2^{n-1}x$.

[*Alld.*, '51; *Luck.*, '66]

[Hint. The nth term of this series may be expressed as $\cot 2^{n-2}x-\cot 2^{n-1}x$.]

2. $\tan\theta+2\tan 2\theta+2^2\tan 2^2\theta+2^3\tan 2^3\theta+\ldots$ to n terms

[*Lucknow*, '59]

3. $\sec\alpha\sec 2\alpha+\sec 2\alpha\sec 3\alpha+\ldots$ to n terms. [*Luck.*, '65]

4. $\tan^{-1}\dfrac{x}{1+1.2x^2}+\tan^{-1}\dfrac{x}{1+2.3x^2}+\ldots$ to n terms.

[*Lucknow*,' 63]

5. $\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{2}{9}+\ldots+\tan^{-1}\dfrac{2^{n-1}}{1+2^{2n-1}}+\ldots$ ad inf.

[*Lucknow*, '62]

6. $\tan^{-1}\dfrac{1}{1+1+1^2}+\tan^{-1}\dfrac{1}{1+2+2^2}+\dots$
$+\tan^{-1}\dfrac{1}{1+n+n^2}.$ [*Lucknow*, '61]

7. $\sin^3\dfrac{\theta}{3}+3\sin^3\dfrac{\theta}{3^2}+3^2\sin^3\dfrac{\theta}{3^3}+\dots$ to n terms.
[*Lucknow*, '65]

8. $\dfrac{\sin\theta}{\cos\theta+\cos 2\theta}+\dfrac{\sin 2\theta}{\cos\theta+\cos 4\theta}+\dfrac{\sin 3\theta}{\cos\theta+\cos 6\theta}+\dots$
to n terms. [*Luck.*, '56]

9. $\dfrac{1}{\cot\theta-3\tan\theta}+\dfrac{3}{\cot 3\theta-3\tan 3\theta}+\dfrac{3^2}{\cot 3^2\theta-3\tan 3^2\theta}$
$+\dots$ to n terms.

10. $\tan\dfrac{x}{2}\sec x+\tan\dfrac{x}{2^2}\sec\dfrac{x}{2}+\tan\dfrac{x}{2^3}\sec\dfrac{x}{2^2}+\dots.$
[*Lucknow*, '58]

11. $\cot^{-1}(2a^{-1}+a)+\cot^{-1}(2a^{-1}+3a)+\cot^{-1}(2a^{-1}+6a)$
$+\cot(2a^{-1}+10a)+\dots$ to n terms.
[*Lucknow*, '56]

12. $\cot^{-1}(2.1^2)+\cot^{-1}(2.2^2)+\cot^{-1}(2.3^2)+\dots$ ad inf.
[*Banaras*, '64]

13. $\cot^{-1}(1^2+\frac{3}{4})+\cot^{-1}(2^2+\frac{3}{4})+\cot^{-1}(3^2+\frac{3}{4})+\dots$ ad inf.
[*Lucknow*, '60]

14. $\tan^2\alpha\tan 2\alpha+\frac{1}{2}\tan^2 2\alpha\tan 4\alpha+\dfrac{1}{2^2}\tan^2 4\alpha\tan 8\alpha$
$+\dots$ to n terms
[*Lucknow*, '61]

15. $2\operatorname{cosec} 2\theta\cot 2\theta+4\operatorname{cosec} 4\theta\cot 4\theta$
$+8\operatorname{cosec} 8\theta\cot 8\theta+\dots$ to n terms.

16. $\cos\dfrac{\theta}{2}+2\cos\dfrac{\theta}{2}\cos\dfrac{\theta}{2^2}+2^2\cos\dfrac{\theta}{2}\cos\dfrac{\theta}{2^2}\cos\dfrac{\theta}{2^3}+\dots$
to n terms.
[*Lucknow*, '64]

17. Prove that

$$\sin\theta \,.\, \text{versin}\,\theta + 2\sin\frac{\theta}{2} \,.\, \text{versin}\,\frac{\theta}{2} + 2^2 \sin\frac{\theta}{2^2}\,\text{versin}\,\frac{\theta}{2^2} + \ldots \,.$$
$$= \theta - \frac{\sin 2\theta}{2},$$

and interpret the result geometrically.

9·9. Use of differentiation and integration. Sometimes it is convenient to obtain the sum of a given series by differentiating or integrating a known series. This method is, however, applicable only when the known series is differentiable or integrable term by term.

Ex. 1. Find the sum of the series

$\cos\theta + 2\cos 2\theta + 3\cos 3\theta + \ldots + n\cos n\theta.$ [*Mysore*, '32]

We know that

$$\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta = \frac{\sin\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta}\sin\tfrac{1}{2}(n+1)\theta.$$

Now differentiating the left-hand side of this, we obtain

$$\cos\theta + 2\cos 2\theta + 3\cos 3\theta + \ldots + n\cos n\theta,$$

and differentiating the right-hand side, we get

$$\begin{aligned}
&\tfrac{1}{2}n\cos\tfrac{1}{2}n\theta \,.\, \sin\tfrac{1}{2}(n+1)\theta \,.\, \text{cosec}\,\tfrac{1}{2}\theta \\
&\quad + \tfrac{1}{2}(n+1)\sin\tfrac{1}{2}n\theta \,.\, \cos\tfrac{1}{2}(n+1)\theta \,.\, \text{cosec}\,\tfrac{1}{2}\theta \\
&\quad - \tfrac{1}{2}\sin\tfrac{1}{2}n\theta\,\sin\tfrac{1}{2}(n+1)\theta \,.\, \text{cosec}\,\tfrac{1}{2}\theta \,.\, \cot\tfrac{1}{2}\theta \\
&= \tfrac{1}{2}n\sin(n+\tfrac{1}{2})\theta\,\text{cosec}\,\tfrac{1}{2}\theta + \tfrac{1}{2}\sin\tfrac{1}{2}n\theta\cos\tfrac{1}{2}(n+1)\theta\,\text{cosec}\,\tfrac{1}{2}\theta \\
&\qquad - \tfrac{1}{2}\sin\tfrac{1}{2}n\theta\sin\tfrac{1}{2}(n+1)\theta\cos\tfrac{1}{2}\theta\,\text{cosec}^2\tfrac{1}{2}\theta \\
&= \tfrac{1}{2}\{n\sin(n+\tfrac{1}{2})\theta\sin\tfrac{1}{2}\theta + \sin\tfrac{1}{2}n\theta\cos\tfrac{1}{2}(n+1)\theta\sin\tfrac{1}{2}\theta \\
&\qquad - \sin\tfrac{1}{2}n\theta\sin\tfrac{1}{2}(n+1)\theta\cos\tfrac{1}{2}\theta\}\,\text{cosec}^2\tfrac{1}{2}\theta \\
&= \tfrac{1}{2}\{n\sin(n+\tfrac{1}{2})\theta\sin\tfrac{1}{2}\theta - \sin^2\tfrac{1}{2}n\theta\}\,\text{cosec}^2\tfrac{1}{2}\theta \\
&= \tfrac{1}{2}[n\{\cos n\theta - \cos(n+1)\theta\} - 1 + \cos n\theta]/(1-\cos\theta) \\
&= \tfrac{1}{2}\{(n+1)\cos n\theta - n\cos(n+1)\theta - 1\}/(1-\cos\theta).
\end{aligned}$$

Hence

$$\cos\theta + 2\cos 2\theta + 3\cos 3\theta + \ldots + n\cos n\theta$$
$$= \tfrac{1}{2}\{(n+1)\cos n\theta - n\cos(n+1)\theta - 1\}/(1-\cos\theta).$$

Ex. 2. Show that the series

$$\cos x+\frac{1}{2.3}\cos 3x+\frac{1.3}{2.4.5}\cos 5x+\ldots\text{ad inf.}$$

has the sum $\sin^{-1}\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)$, when $\pi>x>0$,

and $\qquad -\sin^{-1}\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)$, when $2\pi>x>\pi$.

[*Bombay*, 1947]

First take $\pi>x>0$, and assume that

$$C=\cos x+\tfrac{1}{2}\cos 3x+\frac{1.3}{2.4}\cos 5x+\ldots,$$

and

$$S=\sin x+\tfrac{1}{2}\sin 3x+\frac{1.3}{2.4}\sin 5x+\ldots.$$

Then

$$C+iS=e^{xi}\left\{1+\tfrac{1}{2}e^{2xi}+\frac{1.3}{2.4}e^{4xi}+\ldots\right\}$$

$$=e^{xi}(1-e^{2xi})^{-1/2}$$

$$=e^{xi}(1-e^{-2xi})^{1/2}/(1-e^{2xi})^{1/2}(1-e^{-2xi})^{1/2}$$

$$=\frac{(\cos x+i\sin x)(1-\cos 2x+i\sin 2x)^{1/2}}{\sqrt{(2-2\cos 2x)}}$$

$$=\frac{(\cos x+i\sin x)\sqrt{(2\sin x)}\,.\,(\sin x+i\cos x)^{1/2}}{2\sin x}$$

$$=\frac{(\cos x+i\sin x)\{\cos(\pi/4-x/2)+i\sin(\pi/4-x/2)\}}{\sqrt{(2\sin x)}}$$

$$=\frac{\cos(\pi/4+x/2)+i\sin(\pi/4+x/2)}{\sqrt{(2\sin x)}}.$$

Therefore

$$S\equiv\sin x+\tfrac{1}{2}\sin 3x+\frac{1.3}{2.4}\sin 5x+\ldots$$

$$=\sin(\pi/4+x/2)/\sqrt{(2\sin x)}$$

$$=\frac{\sin(x/2)+\cos(x/2)}{2\sqrt{(\sin x)}}$$

$$=\frac{\tfrac{1}{2}[\sin(x/2)+\cos(x/2)]}{\sqrt{\left\{1-\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2\right\}}}.$$

Integrating this on either side with respect to x, we have

$$\cos x+\frac{1}{2.3}\cos 3x+\frac{1.3}{2.4.5}\cos 5x+\ldots=\sin^{-1}\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)+A,$$

where A is the constant of integration.

Putting $x=\pi/2$, A becomes zero. Therefore, when $\pi>x>0$,

$$\cos x+\frac{1}{2.3}\cos 3x+\frac{1.3}{2.4.5}\cos 5x+\ldots=\sin^{-1}\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right).$$

Again in (1) replacing x by $x-\pi$ we have, when $2\pi>x>\pi$, that

$$\sin x+\tfrac{1}{2}\sin 3x+\frac{1.3}{2.4}\sin 5x+\ldots+=+\frac{\frac{1}{2}[\cos(x/2)-\sin(x/2)]}{\sqrt{\left\{1-\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2\right\}}}.$$

Integrating this on either side with respect to x, we have

$$\cos x+\frac{1}{2.3}\cos 3x+\frac{1.3}{2.4.5}\cos 5x+\ldots=-\sin^{-1}\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right),$$

the constant of integration vanishing as before.

Hence the proposition.

EXAMPLES

1. Given that

$$\cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\ldots\text{ad inf.}=\frac{\sin x}{x^2},$$

prove that

$$\tfrac{1}{2}\tan\frac{x}{2}+\frac{1}{2^2}\tan\frac{x}{2^2}+\frac{1}{2^3}\tan\frac{x}{2^3}+\ldots\text{ad inf.}=\frac{1}{x}-\cot x,$$

and

$$\frac{1}{2^2}\sec^2\frac{x}{2}+\frac{1}{2^4}\sec\frac{x}{2^2}+\frac{1}{2^6}\sec\frac{x}{2^3}+\ldots\text{ad inf.}=\operatorname{cosec}^2 x-\frac{1}{x^2}.$$

2. Sum the series

$$\sin\theta-\tfrac{1}{2}\frac{\sin 3\theta}{3}+\frac{1.3}{2.4}\cdot\frac{\sin 5\theta}{5}-\ldots\text{ad inf.},$$

where $$-\frac{\pi}{2}<\theta<\frac{\pi}{2}.$$

Also find the sum when θ is outside the above limits.

3. Sum the following series by differentiation :
$\tan\theta+2\tan 2\theta+2^2\tan 2^2\theta+\ldots$ to n terms.

4. Sum the following series by integration :
$\operatorname{cosec} x+\operatorname{cosec} 2x+\operatorname{cosec} 4x+\ldots$ to n terms.

Examples on Chapter IX

1. Sum to n terms the series
$\cos\alpha\sin\beta+\cos 3\alpha\sin 2\beta+\cos 5\alpha\sin 3\beta+\ldots$

2. Sum to infinity the series
$$\frac{\cos\theta}{1.2}+\frac{\cos 2\theta}{2.3}+\frac{\cos 3\theta}{3.4}+\ldots.$$

3. Sum the series
$\sin^2\theta\sin 2\theta+\sin^2 2\theta\sin 3\theta+\sin^2 3\theta\sin 4\theta+\ldots$ to n terms.
[*Cal.*, '41]

4. Prove that
$$\sum_0^\infty \frac{n+1}{2!}a^n\cos n\alpha=e^{a\cos\alpha}\{\cos(a\sin\alpha)+a\cos(\alpha+a\sin\alpha)\}.$$
[*Travancore*, '42]

5. Sum to m terms the series
$\sin\theta-\frac{m-1}{2}\sin 2\theta+\frac{(m-1)(m-2)}{2.3}\sin 3\theta-\ldots$ [*Andhra*, '40]

6. Find the sum to infinity of the series
(i) $\frac{c\sin\alpha}{1}-\frac{c^3\sin 3\alpha}{3}+\frac{c^5\sin 5\alpha}{5}-\ldots,$ where $|c|<1$,
(ii) $\frac{\sin\alpha}{1!}+\frac{\sin 5\alpha}{5!}+\frac{\sin 9\alpha}{9!}+\ldots.$

7. Sum the series
$\tan\alpha\tan(\alpha+\beta)+\tan(\alpha+\beta)\tan(\alpha+2\beta)$
$+\tan(\alpha+2\beta)\tan(\alpha+3\beta)+\ldots$ to n terms.
[*Lucknow*, '43]

8. Find the sum to n terms and also the sum to infinity of the series

$$\log\cos\theta + \log\cos(\theta/2) + \log\cos(\theta/2^2) + \ldots$$ [*Alld.*, '51]

9. Sum the series

$$\cos(\pi/3) + \tfrac{1}{2}\cos(2\pi/3) + \tfrac{1}{3}\cos(3\pi/3) + \ldots \text{ad inf.}$$ [*Lucknow*, '62]

10. Find the sum of the series

$$\frac{\cos\theta - \cos 3\theta}{\sin 3\theta} + 3\,\frac{\cos 3\theta - \cos 3^2\theta}{\sin 3^2\theta} + 3^2\frac{\cos 3^2\theta - \cos 3^3\theta}{\sin 3^3\theta} + \ldots \text{to } n \text{ terms.}$$

11. Sum the series

$$\sin^4\alpha + \sin^4(\alpha + 2\pi/n) + \sin^4(\alpha + 4\pi/n) + \ldots + \sin^4\{\alpha + 2(n-1)\pi/n\}.$$ [*Agra*, '44; *Luck.*, '47]

12. Sum the series

$$\frac{1}{\sin\theta\cos 2\theta} - \frac{1}{\cos 2\theta\sin 3\theta} + \frac{1}{\sin 3\theta\cos 4\theta} - \ldots \text{to } n \text{ terms.}$$ [*Agra*, 1941]

13. Sum to infinity the series

$$1 + e^{\cos\theta}\cos(\sin\theta) + \frac{e^{2\cos\theta}}{2!}\cos(2\sin\theta) + \frac{e^{3\cos\theta}}{3!}\cos(3\sin\theta) + \ldots.$$ [*Mysore*, 1941]

14. Sum the series $\sum_1^n \sqrt{(1+\sin rx)}$. [*Cal.*, 1942]

15. Sum the series

$$\tan\theta\sec 2\theta + \tan 2\theta\sec 2^2\theta + \ldots + \tan 2^{n-1}\theta\sec 2^n\theta.$$ [*Lucknow*, 1968]

16. Assuming the formula for the summation

$$\sum_{r=0}^{n-1}\cos(\alpha + r\beta),$$

deduce the sum of each of the following series :

(i) $\sum_{r=0}^{n-1}(-1)^r\cos(\alpha + r\beta)$, (ii) $\sum_{r=0}^{n-1}\cosh(\alpha + r\beta)$. [*Cal.*, 1944]

17. Show how the function $\sin^2 x \cos^4 x$ can be thrown into the form

$$A+B\cos 2x+C\cos 4x+D\cos 6x;$$

and hence or otherwise sum the series

$$\sum_{r=1}^{n} \sin^2 rx \cos^4 rx.$$

18. Reduce the expression

$$\tan^{-1}(2^{n+1})-\tan^{-1}(2^n)$$

to its simplest form, and hence or otherwise sum the infinite series

$$\cot^{-1}(2^2+\tfrac{1}{2})+\cot^{-1}\left(2^3+\frac{1}{2^2}\right)+\cot^{-1}\left(2^4+\frac{1}{2^3}\right)+$$

$$\ldots+\cot^{-1}\left(2^{n+1}+\frac{1}{2^n}\right)+\ldots.$$ [*Gauhati*, 50'; *Luck.*, '63]

19. Sum the series

$$\sin\alpha\sin\beta+\tfrac{1}{2}\sin 2\alpha\sin 2\beta+\tfrac{1}{3}\sin 3\alpha\sin 3\beta+\ldots\text{ad inf.}$$

[*Lucknow*, '57]

20. Find the sum to infinity of the series

$$\sin\theta\cos\phi+\tfrac{1}{3}\sin 3\theta\cos 3\phi+\tfrac{1}{5}\sin 5\theta\cos 5\phi+\ldots,$$

where θ and ϕ are positive acute angles.

[*Travancore*, '51]

21. Sum the series $(-\pi/4<\theta<\pi/4)$

$$2\tan\theta-\tfrac{4}{3}\tan^3\theta+\tfrac{6}{5}\tan^5\theta-\tfrac{8}{7}\tan^7\theta+\ldots.$$

[*Calcutta*, '42]

22. Sum the series

$$\sin 2\theta\cos^2\theta-\tfrac{1}{2}\sin 4\theta\cos^2 2\theta+\tfrac{1}{4}\sin 8\theta\cos^2 4\theta+\ldots \text{ to } n \text{ terms.}$$

[*Punjab*, '44; *Banaras*, '51]

23. Sum the series

$$1-\frac{\cos 2\theta}{2!}+\frac{\cos 4\theta}{4!}-\frac{\cos 6\theta}{6!}+\ldots.$$ [*Dacca*, '40]

24. Sum the series

$$\cos^2 x-\tfrac{1}{2}\sin^2 2x+\tfrac{1}{3}\cos^2 3x-\tfrac{1}{4}\sin^2 4x+\ldots$$ [*Lucknow*, '65]

25. Sum to n terms

$$\frac{\tan^3\theta}{1-3\tan^2\theta}+\left(\frac{1}{3}\right)\frac{\tan^3 3\theta}{1-3\tan^2 3\theta}+\left(\frac{1}{3^2}\right)\frac{\tan^3 3^2\theta}{1-3\tan^2 3^2\theta}+\ldots$$

[*Panjab*, '45]

26. Sum the series

$$\frac{1}{2^2}\tan\frac{\pi}{2^2}+\frac{1}{2^3}\tan\frac{\pi}{2^3}+\frac{1}{2^4}\tan\frac{\pi}{2^4}+\ldots$$

27. Prove that

$$\sin\theta+\tfrac{1}{3}\sin^3\theta+\tfrac{1}{5}\sin^5\theta+\ldots$$
$$=2(\sin\theta-\tfrac{1}{3}\sin 3\theta+\tfrac{1}{5}\sin 5\theta-\ldots),$$

where $\theta\neq\frac{1}{2}(2n+1)\pi$.

28. Prove that, if $\tan\theta<1$,

$$\tan^2\theta-\tfrac{1}{2}\tan^4\theta+\tfrac{1}{3}\tan^6\theta-\ldots$$
$$=\sin^2\theta+\tfrac{1}{2}\sin^4\theta+\tfrac{1}{3}\sin^6\theta+\ldots$$

29. If S_n be the sum to n terms of the series

$$\sin x+\sin 2x+\sin 3x+\ldots,$$

prove that

$\lim_{n\to\infty}(S_1+S_2+S_3+\ldots+S_n)/n=\frac{1}{2}\cot\frac{1}{2}x.$ [*Lucknow*, 1967]

30. $A_1A_2\ldots A_n$ is a regular polygon of n sides inscribed in a circle, whose centre is O, and P is any point on the arc A_nA_1 such that the angle POA_1 is θ; find the sum of the lengths of the lines joining P to the angular points of the polygon.

31. From any point on the circumference of a circle of radius r chords are drawn to the angular points of a regular inscribed polygon of n sides. Show that the sum of the squares of the chords is $2r^2n$. [*Annam.*, '43]

32. A regular polygon of n sides is circumscribed about a circle of radius a. The distances of a point P within the polygon from the sides are $p_1, p_2, p_3, \ldots, p_n$, and from the centre the distance is d. Show that

$$p_1+p_2+p_3+\ldots+p_n=na,$$

and $p_1^2+p_2^2+p_3^3+\ldots+p_n^2=n(a^2+\tfrac{1}{2}d^2)$. [*Annam.*, '39]

CHAPTER X

FACTORIZATION. INFINITE PRODUCTS

10·1. Factorization. We know that if $f(x)$ be any polynomial and if $x=a$ be a root of the equation $f(x)=0$, then $(x-a)$ is a factor of $f(x)$. Therefore, in order to get the factors of any polynomial $f(x)$ we solve the equation $f(x)=0$. If $a_1, a_2, ..., a_n$ be the roots of the equation $f(x)=0$, of degree n in x, then $(x-a_1), (x-a_2), ..., (x-a_n)$ are the n factors of $f(x)$. If the coefficient of the highest degree term in x be unity, then

$$f(x)=(x-a_1)(x-a_2)...(x-a_n).$$

We shall make use of this principle in the following sections.

10·2. Factorization of x^n-1.

Solving $x^n-1=0$, we have (see § 3·2)

$$x=\cos(2r\pi/n)+i\sin(2r\pi/n),$$

where $r=0, 1, 2, 3, ..., (n-1)$.

Thus the roots of $x^n-1=0$ are

$1, \cos(2\pi/n)+i\sin(2\pi/n), \cos(4\pi/n)+i\sin(4\pi/n),$
$..., \cos(4\pi/n)-i\sin(4\pi/n), \cos(2\pi/n)-i\sin(2\pi/n).$

It will be noted that the nth root is conjugate to the second, the $(n-1)$th root is conjugate to the third, and so on.

Two cases now arise.

CASE I. Let n be even.

In this case there are two real roots 1 and -1, which correspond to $r=0$ and $r=\frac{1}{2}n$, and $\frac{1}{2}n-1$ pairs of conjugate imaginary roots, viz.

$$\cos(2\pi/n) \pm i \sin(2\pi/n),\ \cos(4\pi/n) \pm i \sin(4\pi/n),\ \ldots,\ \cos\{(n-2)\pi/n\} \pm i \sin\{(n-2)\pi/n\}.$$

Consequently, the factors of x^n-1 are

$$(x-1),\ (x+1);\ [x-\{\cos(2\pi/n)+i\sin(2\pi/n)\}],$$
$$[x-\{\cos(2\pi/n)-i\sin(2\pi/n)\}];\ \ldots;$$
$$[x-\{\cos(n-2)\pi/n+i\sin(n-2)\pi/n\}]$$
$$[x-\{\cos(n-2)\pi/n-i\sin(n-2)\pi/n\}];$$

or

$$(x^2-1),\ [x^2-2x\cos(2\pi/n)+1],\ \ldots,\ [x^2-2x\cos\{(n-2)\pi/n\}+1].$$

Hence, *when n is even,*

$$x^2-1=(x^2-1)[x^2-2x\cos(2\pi/n)+1]$$
$$\times[x^2-2x\cos(4\pi/n)+1]\ldots,$$
$$\times[x^2-2x\cos\{(n-2)\pi/n\}+1]$$

$$=(x^2-1)\prod_{r=1}^{(n/2)-1}\{x^2-2x\cos(2r\pi/n)+1\}. \qquad (1)$$

The symbol

$$\prod_{r=1}^{(n/2)-1}(x^2-2x\cos(2r\pi/n)+1\}$$

denotes the products of factors of the type

$$\{x^2-2x\cos(2r\pi/n)+1\},$$

which are obtained by replacing r by integers beginning with 1 and ending in $\frac{1}{2}n-1$.

CASE II. Let n be odd.

In this case there is only one real root, viz. 1, corresponding to $r=0$, and $\frac{1}{2}(n-1)$ pairs of conjugate imaginary roots, viz.

$$\cos(2\pi/n) \pm i \sin(2\pi/n),\ \cos(4\pi/n) \pm i \sin(4\pi/n),\ \ldots,\ \cos\{(n-1)\pi/n\} \pm i \sin\{(n-1)\pi/n\}.$$

Consequently, the factors of x^n-1 are

$$(x-1);\ [x-\{\cos(2\pi/n + i\sin(2\pi/n)\}],$$
$$[x-\{\cos(2\pi/n) - i\sin(2\pi/n)\}];\ldots;$$
$$[x-\{\cos(n-1)\pi/n + i\sin(n-1)\pi/n\}],$$
$$[x-\{\cos(n-1)\pi/n - i\sin(n-1)\pi/n\}],$$

or

$$(x-1),\ [x^2-2x\cos(2\pi/n)+1],\ [x^2-2x\cos(4\pi/n)+1],\ \ldots,\ [x^2-2x\cos\{(n-1)\pi/n\}+1].$$

Hence, *when n is odd*,

$$x^n-1 = (x-1)[x^2-2x\cos(2\pi/n)+1] \times [x^2-2x\cos(4\pi/n)+1]\ldots \times [x^2-2x\cos\{(n-1)\pi/n\}+1]$$

$$= (x-1)\prod_{r=1}^{(n-1)/2}\{x^2-2x\cos(2r\pi/n)+1\}. \qquad (2)$$

10·3. Factorization of x^n+1.

Solving $x^n+1=0$, we have

$$x = \cos\{(2r+1)\pi/n)\} + i\sin\{(2r+1)\pi/n\},$$

where $r = 0, 1, 2, \ldots, (n-1)$.

Thus the roots of $x^n+1=0$ are

$$\cos(\pi/n)+i\sin(\pi/n),\ \cos(3\pi/n)+i\sin(3\pi/n),\ \ldots,\ \cos(3\pi/n)-i\sin(3\pi/n),\ \cos(\pi/n)-i\sin(\pi/n).$$

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Evidently the nth root is conjugate to the first, the $(n-1)$th root is conjugate to the second, and so on.

Two cases now arise.

CASE I. Let n be even.

In this case, there are $\frac{1}{2}n$ pairs of conjugate imaginary roots, viz.

$$\cos(\pi/n) \pm i \sin(\pi/n),\ \cos(3\pi/n) \pm i \sin(3\pi/n),\ \ldots,\ \cos\{(n-1)\pi/n\} \pm i \sin\{(n-1)\pi/n\}.$$

Consequently, the factors of x^n+1 are

$$[x-\{\cos(\pi/n)+i\sin(\pi/n)\}],\quad [x-\{\cos(\pi/n)-i\sin(\pi/n)\}];$$
$$[x-\{\cos(3\pi/n]+i\sin(3\pi/n)\}],\quad [x-\{\cos(3\pi/n)-i\sin(3\pi/n)\}];$$
$$\ldots \qquad \ldots \qquad \ldots$$
$$[x-\cos\{(n-1)\pi/n\}-i\sin\{(n-1)\pi/n\}],\quad [x-\cos\{(n-1)\pi/n\}+i\sin\{(n-1)\pi/n\}];$$

or,
$$[x^2-2x\cos(\pi/n)+1],\ [x^2-2x\cos(3\pi/n)+1],\ \ldots,\ [x^2-2x\cos\{(n-1)\pi/n\}+1].$$

Hence, *when n is even,*

$$x^n+1=[x^2-2x\cos(\pi/n)+1][x^2-2x\cos(3\pi/n)+1]\ldots\ldots[x^2-2x\cos\{(n-1)\pi/n\}+1]$$

$$=\prod_{r=0}^{n/2-1}[x^2-2x\cos\{(2r+1)\pi/n\}+1]. \qquad (3)$$

CASE II Let n be odd.

In this case, there is one real root, viz, -1, corresponding to $r=\frac{1}{2}(n-1)$, and $\frac{1}{2}(n-1)$ pairs of conjugate imaginary roots, viz.

$$\cos(\pi/n) \pm i \sin(\pi/n),\ \cos(3\pi/n) \pm i \sin(3\pi/n),\ \ldots,$$
$$\cos\{(n-2)\pi/n\} \pm i \sin\{(n-2)\pi/n\}.$$

Consequently, the factors of x^n+1 are

$$(x+1);\ [x-\{\cos(\pi/n)+i \sin(\pi/n)\}],$$
$$[x-\{\cos(\pi/n)-i \sin(\pi/n)\}],$$
$$[x-\{\cos(3\pi/n)+i \sin(3\pi/n)\}],$$
$$[x-\{\cos(3\pi/n)-i \sin(3\pi/n)\}];$$
$$\ldots \qquad \ldots \qquad \ldots$$
$$[x-\cos\{(n-2)\pi/n\}-i \sin\{(n-2)\pi/n\}],$$
$$[x-\cos\{(n-2)\pi/n\}+i \sin\{(n-2)\pi/n\}],$$

or

$$(x+1),\ [x^2-2x \cos(\pi/n)+1],\ \ldots,$$
$$[x^2-2x \cos\{(n-2)\pi/n)\}+1].$$

Hence, *when n is odd*,

$$x^n+1=(x+1)[x^2-2x \cos(\pi/n)+1]\ldots$$
$$[x^2-2x \cos\{(n-2)\pi/n\}+1]$$
$$=(x+1) \prod_{r=0}^{(n-3)/2} [x^2-2x \cos\{(2r+1)\pi/n\}+1]. \qquad (4)$$

10·4. Factorization of $x^{2n}-2x^n \cos n\theta+1$.

Solving the equation $x^{2n}-2x^n \cos n\theta+1=0$, we have

$$x^n=\cos n\theta \pm \sqrt{(\cos^2 n\theta-1)}$$
$$=\cos n\theta \pm i \sin n\theta$$
$$=\cos(n\theta+2r\pi) \pm i \sin(n\theta+2r\pi),$$

or $\quad x=\cos(\theta+2r\pi/n) \pm i \sin(\theta+2r\pi/n)$,

where $r=0, 1, 2, 3, \ldots, (n-1)$.

Thus the roots of $x^{2n}-2x^n \cos n\theta+1=0$ are $\cos\theta \pm i\sin\theta$, $\cos(\theta+2\pi/n)\pm i\sin(\theta+2\pi/n)$, $\cos(\theta+4\pi/n)\pm i\sin(\theta+4\pi/n),\ldots$, $\cos\{\theta+2(n-1)\pi/n\}\pm i\sin\{\theta+2(n-1)\pi/n\}$.

Thus the factors of the given expression are $[x^2-2x\cos\theta+1]$, $[x^2-2x\cos(\theta+2\pi/n)+1]$, ..., $[x^2-2x\cos\{\theta+2(n-1)\pi/n\}+1]$.

Hence, we finally have

$$\begin{aligned} x^{2n}-2x^n\cos n\theta+1 &= [x^2-2x\cos\theta+1] \\ &\quad\times[x^2-2x\cos(\theta+2\pi/n)+1]\ldots \\ &\quad\times[x^2-2x\cos\{\theta+2(n-1)\pi/n)\}+1] \\ &= \prod_{r=0}^{n-1}[x^2-2x\cos(\theta+2r\pi/n)+1]. \qquad (5) \end{aligned}$$

10·41. Deductions. Several important results may be deduced from (5). For example, dividing each side of (5) by x^n, we get

$$x^n+\frac{1}{x^n}-2\cos n\theta=\prod_{r=0}^{n-1}\left[x+\frac{1}{x}-2\cos(\theta+2r\pi/n)\right], \qquad (6)$$

and further putting $\theta=0$, we have

$$x^n+\frac{1}{x^n}-2=\prod_{r=0}^{n-1}\left[x+\frac{1}{x}-2\cos(2r\pi/n)\right]. \qquad (7)$$

Again in (5) replacing x by x/a and multiplying each side by a^{2n}, we get

$$x^{2n}-2a^nx^n\cos n\theta+a^{2n}=\prod_{r=0}^{n-1}[x^2-2ax\cos(\theta+2r\pi/n)+a^2], \qquad (8)$$

and further putting $\theta=0$, we have

$$x^{2n}-2a^nx^n+a^{2n}=\prod_{r=0}^{n-1}[x^2-2ax\cos(2r\pi/n)+a^2]. \qquad (9)$$

The results of § 10·2 and § 10·3 may also be deduced from (5). Thus putting $\theta=0$ in (5), we get

$$(x^n-1)^2=\prod_{r=0}^{n-1}[x^2-2x\cos(2r\pi/n)+1],$$

which, on extracting the square root, reduces to

$$x^n-1=(x^2-1)\prod_{r=1}^{n/2-1}(x^2-2x\cos(2r\pi/n)+1),$$

when n is even; and to

$$x^n-1=(x-1)\prod_{r=1}^{(n-1)/2}[x^2-2x\cos(2r\pi/n)+1],$$

when n is odd.

Similarly, putting $\theta=\pi/n$ in (5), we get

$$(x^n+1)^2=\prod_{r=0}^{n-1}[x^2-2x\cos\{(2r+1)\pi/n\}+1],$$

which, on extracting the square root, reduces to

$$x^n+1=\prod_{r=0}^{n/2-1}[x^2-2x\cos\{(2r+1)\pi/n\}+1],$$

when n is even; and to

$$x^n+1=(x+1)\prod_{r=0}^{(n-3)/2}[x^2-2x\cos\{(2r+1)\pi/n\}+1]$$

when n is odd.

Ex. 1. Factorize $\cos n\phi-\cos n\theta$. [*Annamalai*, 1939]

In (6) putting $x=e^{\phi i}$, so that $\frac{1}{x}=e^{-\phi i}$, we have

$$e^{n\phi i}+e^{-n\phi i}-2\cos n\theta=\prod_{r=0}^{n-1}[e^{\phi i}+e^{-\phi i}-2\cos(\theta+2r\pi/n)]$$

or $$2\cos n\phi-2\cos n\theta=\prod_{r=0}^{n-1}[2\cos\phi-2\cos(\theta+2r\pi/n)].$$

Therefore,

$$\cos n\phi-\cos n\theta=2^{n-1}\prod_{r=0}^{n-1}[\cos\phi-\cos(\theta+2r\pi/n)].$$

Ex. 2. Show that

$$2^{n-1}\sin\phi\sin(\phi+2\pi/n)\sin(\phi+4\pi/n)\ldots\sin\{\phi+(2n-2)\pi/n\}$$
$$=\cos(\pi/2)-\cos n(\phi+\pi/2).$$

In Ex. 1 above putting $\phi=\frac{1}{2}\pi$ and $\theta=\phi+\frac{1}{2}\pi$, we have

$$\cos(\tfrac{1}{2}n\pi)-\cos n(\phi+\tfrac{1}{2}\pi)=2^{n-1}\prod_{r=0}^{n-1}[\cos(\tfrac{1}{2}\pi)-\cos(\tfrac{1}{2}\pi+\phi+2r\pi/n)]$$

$$=2^{n-1}\prod_{r=0}^{n-1}\sin(\phi+2r\pi/n)$$

$$=2^{n-1}\sin\phi\sin(\phi+2\pi/n)\ldots\sin\{\phi+(2n-2)\pi/n\}.$$

Ex. 3. Show that

$$\sin n\phi=2^{n-1}\sin\phi\sin\left(\phi+\frac{\pi}{n}\right)\ldots\sin\left(\phi+\frac{n-1}{n}\pi\right).$$

[*Travancore*, 1951]

In (6) putting $x=1$ and $\theta=2\phi$, we get

$$2-2\cos 2n\phi=\prod_{r=0}^{n-1}[2-2\cos(2\phi+2r\pi/n)].$$

or

$$2^2\sin^2 n\phi=\prod_{r=0}^{n-1}[2^2\sin^2(\phi+r\pi/n)].$$

$\therefore$

$$\sin n\phi=2^{n-1}\prod_{r=0}^{n-1}\sin(\phi+r\pi/n)$$

$$=2^{n-1}\sin\phi\sin(\phi+\pi/n)\ldots\sin\{\phi+(n-1)\pi/n\}.$$

Ex. 4. Prove that

$$2^{(n-1)/2}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{(n-1)\pi}{2n}=\sqrt{n}.$$

[*Travancore*, 1951]

From Ex. 3, we have

$$\frac{\sin n\phi}{\sin\phi}=2^{n-1}\sin\left(\phi+\frac{\pi}{n}\right)\sin\left(\phi+\frac{2\pi}{n}\right)\ldots\sin\left(\phi+\frac{(n-1)\pi}{n}\right).$$

Making ϕ tend to zero, we have

$$n=2^{n-1}\sin(\pi/n)\sin(2\pi/n)\ldots\sin\{(n-1)\pi/n\}, \qquad \text{(i)}$$

because
$$\lim_{\phi\to 0}\frac{\sin n\phi}{\sin n}=n.$$

Now in the right-hand side of (i), factors equidistant from the beginning and end are the same, so that taking the square root of either side of (i) we have

$$2^{(n-1)/2}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\ldots\sin\frac{(n-1)\pi}{2n}=\sqrt{n}.$$

Ex. 5. Show that

$$(x+a)^{2n}-(x-a)^{2n}=4nxa\prod_{r=1}^{n-1}\left(x^2+a^2\cot^2\frac{r\pi}{2n}\right).$$

[*Annamalai*, 1939]

From (1), we have, on replacing x by y and n by $2n$,

$$y^{2n}-1=(y^2-1)\prod_{r=1}^{n-1}\{y^2-2y\cos(r\pi/n)+1\}.$$

Next replacing y by $(x+a)/(x-a)$ and then multiplying throughout by $(x-a)^{2n}$, we have

$$(x+a)^{2n}-(x-a)^{2n}=\{(x+a)^2-(x-a)^2\}\prod_{r=1}^{n-1}[(x+a)^2$$
$$-2(x^2-a^2)\cos(r\pi/n)+(x-a)^2]$$

$$=4xa\prod_{r=1}^{n-1}2\left[x^2\left(1-\cos\frac{r\pi}{n}\right)+a^2\left(1+\cos\frac{r\pi}{n}\right)\right]$$

$$=4xa\prod_{r=1}^{n-1}4\left[x^2\sin^2\frac{r\pi}{2n}+a^2\cos^2\frac{r\pi}{2n}\right]. \qquad \text{(i)}$$

Now dividing both sides by x^{2n-1} and then making x infinitely large, we get

$$4na=4a\prod_{r=1}^{n-1}4\sin^2\frac{r\pi}{2n}. \qquad \text{(ii)}$$

Dividing (i) by (ii) and multiplying both sides by $4na$ we get the desired result.

EXAMPLES

1. Investigate the complete *trigonometrical* solution of the equation $x^5-1=0$, and hence or otherwise establish the formula of factorization :—

$$x^5-1=(x-1)(x^2-2x\cos\frac{2\pi}{5}+1)(x^2+2x\cos\frac{\pi}{5}+1).$$

Finally assign a suitable numerical value to x so as to deduce the *simplest* numerical value of

$$\sin\frac{\pi}{5}\cos\frac{\pi}{10}.$$ [*Cal.*, 1949]

2. Factorize x^8-1. [*Annam.*, 1947]

3. Resolve x^7+1 into real and quadratic factors. Deduce that

$$\cos\frac{\pi}{7}\cos\frac{3\pi}{7}\cos\frac{5\pi}{7}=-\tfrac{1}{8}.$$ [*Travancore*, 1944]

4. Find the real quadratic factors of

$$x^{12}+1=0.$$

5. Prove that

$$\cosh n\phi-\cos n\theta=2^{n-1}\prod_{s=0}^{n-1}\left\{\cosh\phi-\cos\left(\theta+\frac{2s\pi}{n}\right)\right\}.$$

[*Utkal*, 1950; *Delhi*, 1953]

6. Prove that

(i) $$\cos n\theta=2^{n-1}\left(\cos\theta-\cos\frac{\pi}{2n}\right)\left(\cos\theta-\cos\frac{3\pi}{2n}\right)\cdots\left(\cos\theta-\cos\frac{2n-1}{2n}\pi\right).$$

(ii) $$\cos n\theta=2^{n-1}\sin\left(\theta+\frac{\pi}{2n}\right)\sin\left(\theta+\frac{3\pi}{2n}\right)\cdots\sin\left(\theta+\frac{2n-1}{2n}\pi\right).$$

7. Prove that

$$\sin 2n\theta=2^{2n-1}\sin\theta\cos\theta\prod_{r=1}^{n-1}\left(\cos^2\theta-\cos^2\frac{r\pi}{2n}\right).$$

Hence show that

$$\prod_{r=1}^{n-1} \sin \frac{r\pi}{2n} = \sqrt{n}/2^{n-1}. \qquad \text{[\textit{Madras}, 1938]}$$

8. If n be even, prove that

(i) $2^{(n-1)/2} \sin \frac{2\pi}{2n} \sin \frac{4\pi}{2n} \sin \frac{6\pi}{2n} \dots \sin \frac{n-2}{2n}\pi = \sqrt{n}.$

[*Cal.*, 1943]

(ii) $2^{(n-1)/2} \sin \frac{\pi}{2n} \sin \frac{3\pi}{2n} \sin \frac{5\pi}{2n} \dots \sin \frac{n-1}{2n}\pi = 1.$

[*Patna*, 1935]

(iii) $2^{(n-1)/2} \cos \frac{\pi}{2n} \cos \frac{3\pi}{2n} \cos \frac{5\pi}{2n} \dots \cos \frac{n-1}{2n}\pi = 1.$

(iv) $\tan \psi \tan \left(\psi + \frac{\pi}{n}\right) \tan \left(\psi + \frac{2\pi}{n}\right) \dots$ to n factors

$= (-1)^{n/2}.$ [*Cal.*, 1938]

9. Show that

$$2^{n-1/2} \sin \frac{\pi}{4n} \sin \frac{3\pi}{4n} \dots \sin \frac{2n-1}{4n} = 1. \qquad \text{[\textit{Annam.}, '47]}$$

10. Show that

$$2^{n-1} \cos \frac{\pi}{n} \cos \frac{2\pi}{n} \dots \cos \frac{n-1}{n}\pi = 0,\ 1 \text{ or } -1,$$

according as n is even or of the form $4p+1$ or $4p-1$.

[*Calcutta*, 1940]

11. Prove that

$$\sin \frac{\pi}{n} \sin \frac{2\pi}{n} \dots \sin \frac{n-1}{n}\pi = \frac{n}{2^{n-1}} \qquad \text{[\textit{Cal.}, 1946]}$$

[This is result (i) of solved Ex. 4.]

12. If n be odd, prove that

(i) $2^{(n-1)/2} \sin \frac{2\pi}{2n} \sin \frac{4\pi}{2n} \dots \sin \frac{n-1}{2n}\pi = \sqrt{n}.$

(ii) $2^{(n-1)/2} \cos \frac{\pi}{2n} \cos \frac{3\pi}{2n} \dots \cos \frac{n-2}{2n}\pi = \sqrt{n}.$

(iii) $\tan n\theta = (-1)^{(n-1)/2} \tan\theta \tan\left(\theta + \dfrac{\pi}{n}\right) \ldots$

$$\tan\left(\theta + \frac{n-1}{n}\pi\right).$$

(iv) $\tan\dfrac{\pi}{n}\tan\dfrac{2\pi}{n}\ldots\tan\dfrac{\frac{1}{2}(n-1)\pi}{n} = \sqrt{n}.$ [*Bom.*, 1931]

13. Prove that

$$\frac{(1+x)^n - (1-x)^n}{2x} = A\left(x^2 + \tan^2\frac{\pi}{n}\right)\left(x^2 + \tan^2\frac{2\pi}{n}\right)\ldots$$
$$\left(x^2 + \tan^2\frac{r\pi}{n}\right),$$

where $r = \frac{1}{2}(n-1)$ or $\frac{1}{2}n - 1$, and A is 1 or n, according as n is odd or even. [*Bombay*, 1947]

14. If $A_1A_2A_3\ldots A_n$ be a regular polygon of n sides inscribed in a circle of radius a and P any point in the plane of the circle at a distance x from O, the centre of the circle, show that

$$PA_1^2 \,.\, PA_2^2 \,.\, PA_3^2 \ldots PA_n^2 = x^{2n} - 2x^n a^n \cos n\theta + a^{2n},$$

where $\angle POA_1 = \theta$. [*Delhi*, 1951]

Discuss the cases when $\theta = 0$, or π/n, and $x = a$.

15. A regular polygon $A_1A_2A_3\ldots A_{2m}$ has $2m$ sides. Show that the product of the perpendiculars from the centre of the circumscribed circle on the sides A_1A_2, A_1A_3, $A_1A_4, \ldots, A_1A_m$ is $(\frac{1}{2}a)^{m-1}\sqrt{m}$, where a is the circum-radius. [*Andhra*, 1940]

16. If D_1, D_2, $D_3 \ldots$, D_n be the distances of the vertices of a regular polygon of n sides from any point P in its plane, prove that

$$\frac{1}{D_1^2} + \frac{1}{D_2^2} + \ldots + \frac{1}{D_n^2} = \frac{n}{r^2 - a^2} \times \frac{r^{2n} - a^{2n}}{r^{2n} - 2a^n r^n \cos n\theta + a^{2n}},$$

where a is the radius of the circumcircle of the polygon, r the distance of P from its centre O, and θ the angle that OP makes with the radius to any angular point of the polygon. [*Annam.*, 1951]

INFINITE PRODUCTS

10·5. To express sin θ as an infinite product. We have

$$\sin\theta = 2\sin(\tfrac{1}{2}\theta)\cos(\tfrac{1}{2}\theta)$$
$$= 2\sin(\tfrac{1}{2}\theta)\sin\tfrac{1}{2}(\pi+\theta). \qquad (1)$$

Replacing θ in (1) successively by $\frac{1}{2}\theta$ and $\frac{1}{2}(\theta+\pi)$ we have

$$\sin\tfrac{1}{2}\theta = 2\sin(\theta/2^2)\sin\{(2\pi+\theta)/2^2\}$$

and $\sin\frac{1}{2}(\pi+\theta) = 2\sin\{(\pi+\theta)/2^2\}\sin\{(3\pi+\theta)/2^2\}$.

Substituting these values in the right-hand side of (1), we get after rearranging

$$\sin\theta = 2^3\sin(\theta/2^2)\sin\{(\pi+\theta)/2^2\}\sin\{(2\pi+\theta)/2^2\}$$
$$\sin\{(3\pi+\theta)/2^2\}.$$

Continuing the above process successively, we get

$$\sin\theta = 2^7\sin(\theta/2^3)\sin\{(\pi+\theta)/2^3\}\ldots\sin\{(7\pi+\theta)/2^3\}$$
$$= \ldots \qquad \ldots \qquad \ldots$$
$$= 2^{p-1}\sin(\theta/p)\sin\{(\pi+\theta)/p\}\ldots\sin[\{(p-1)\pi+\theta\}/p], \quad . \quad . \quad . \quad (2)$$

where p is a power of 2.

The last factor on the right-hand side of (2)

$$= \sin\frac{(p-1)\pi+\theta}{p} = \sin\left(\pi - \frac{\pi-\theta}{p}\right) = \sin\frac{\pi-\theta}{p}.$$

The last but one factor

$$= \sin\frac{(p-2)\pi+\theta}{p} = \sin\left[\pi - \frac{2\pi-\theta}{p}\right] = \sin\frac{2\pi-\theta}{p},$$

and so on.

Hence taking together the second and the last factors, the third and the last but one, and so on, (2) becomes

$$\sin\theta = 2^{p-1}\sin(\theta/p)\left\{\sin\frac{\pi+\theta}{p}\sin\frac{\pi-\theta}{p}\right\}$$
$$\times\left\{\sin\frac{2\pi+\theta}{p}\sin\frac{2\pi-\theta}{p}\right\}\cdots$$
$$\times\left\{\sin\frac{(\frac{1}{2}p-1)\pi+\theta}{p}\sin\frac{(\frac{1}{2}p-1)\pi-\theta}{p}\right\}\sin\frac{\frac{1}{2}p\pi+\theta}{p}$$
$$=2^{p-1}\sin(\theta/p)\left[\sin^2\frac{\pi}{p}-\sin^2\frac{\theta}{p}\right]\left[\sin^2\frac{2\pi}{p}-\sin^2\frac{\theta}{p}\right]\cdots$$
$$\times\left[\sin^2\frac{(\frac{1}{2}p-1)\pi}{p}-\sin^2\frac{\theta}{p}\right]\cos\frac{\theta}{p}. \qquad (3)$$

Now divide both sides of (3) by $\sin(\theta/p)$ and make θ tend to zero. Then since

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1,\ \lim_{\theta\to 0}\sin^2(\theta/p)=0 \text{ and}$$

$$\lim_{\theta\to 0}\frac{\sin\theta}{\sin(\theta/p)}=\lim_{\theta\to 0}p\left\{\frac{\sin\theta}{\theta}\right\}\left\{\frac{\theta/p}{\sin(\theta/p)}\right\}=p,$$

we have

$$p=2^{p-1}\sin^2\frac{\pi}{p}\sin^2\frac{2\pi}{p}\sin^2\frac{3\pi}{p}\sin^2\frac{(\frac{1}{2}p-1)\pi}{p}. \qquad (4)$$

Dividing (3) by (4), we have

$$\sin\theta=p\sin\frac{\theta}{p}\cos\frac{\theta}{p}\left[1-\frac{\sin^2(\theta/p)}{\sin^2(\pi/p)}\right]\left[1-\frac{\sin^2(\theta/p)}{\sin^2(2\pi/p)}\right]\cdots$$
$$\left[1-\frac{\sin^2(\theta/p)}{\sin^2\{(p/2-1)\pi/p\}}\right].$$

Now make p tend to infinity. Then since

$$\lim_{p\to\infty}\left(p\sin\frac{\theta}{p}\right)=\lim_{p\to\infty}\left(\frac{\sin(\theta/p)}{\theta/p}\right).\theta=\theta,$$

$$\lim_{p\to\infty}\frac{\sin^2(\theta/p)}{\sin^2(\pi/p)}=\lim_{p\to\infty}\left[\frac{\sin^2(\theta/p)}{\theta^2/p^2}\right]\left[\frac{\pi^2/p^2}{\sin^2(\pi/p)}\right]\left[\frac{\theta^2}{\pi^2}\right]$$

$$=\frac{\theta^2}{\pi^2},$$

and so on; we have

$$\sin\theta=\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\cdots$$

$$=\theta\prod_{r=1}^{\infty}\left(1-\frac{\theta^2}{r^2\pi^2}\right). \tag{5}$$

10·51. Alternative Method.

If in the result (1) of § 10·2 we write $2k$ for n and then divide throughout by x^k, we get

$$x^k-x^{-k}=(x-x^{-1})\prod_{r=1}^{k-1}\left\{x+x^{-1}-2\cos\frac{r\pi}{k}\right\}.$$

Putting $x=e^{\theta i/k}$, we have

$$e^{\theta i}-e^{-\theta i}=(e^{\theta i/k}-e^{-\theta i/k})\prod_{r=1}^{k-1}\left\{e^{\theta i/k}+e^{-\theta i/k}-2\cos\frac{r\pi}{k}\right\}$$

or $$2i\sin\theta=2i\sin(\theta/k)\prod_{r=1}^{k-1}\left\{2\cos\frac{\theta}{k}-2\cos\frac{r\pi}{k}\right\}$$

i.e., $$\frac{\sin\theta}{\sin(\theta/k)}=2^{2(k-1)}\prod_{r=1}^{k-1}\left(\sin^2\frac{r\pi}{2k}-\sin^2\frac{\theta}{2k}\right). \tag{1}$$

Making θ tend to zero, we have

$$k=2^{2(k-1)}\prod_{r=1}^{k-1}\sin^2\frac{r\pi}{2k}. \tag{2}$$

Dividing (1) by (2), we have

$$\frac{\sin\theta}{k\sin(\theta/k)}=\prod_{r=1}^{k-1}\left\{1-\frac{\sin^2(\theta/2k)}{\sin^2(r\pi/2k)}\right\}.$$

Now make k tend to infinity. Then

$$\sin\theta = \theta \prod_{r=1}^{\infty}\left(1-\frac{\theta^2}{r^2\pi^2}\right).$$

10·6. To express $\cos\theta$ as an infinite product.

If in the result (2) of § 10·5, we replace θ by $\frac{1}{2}\pi+\theta$, we get

$$\cos\theta = 2^{p-1}\sin\frac{\pi+2\theta}{2p}\sin\frac{3\pi+2\theta}{2p}\ldots\sin\frac{(2p-1)\pi+2\theta}{2p}.$$

The last factor on the right-hand side

$$=\sin\frac{(2p-1)\pi+2\theta}{2p}=\sin\left\{\pi-\frac{\pi-2\theta}{2p}\right\}$$

$$=\sin\frac{\pi-2\theta}{2p};$$

the last but one factor

$$=\sin\frac{(2p-3)\pi+2\theta}{2p}=\sin\frac{3\pi-2\theta}{2p};$$

and so on.

Now combining together the factors equidistant from the beginning and end, we have

$$\cos\theta = 2^{p-1}\left\{\sin\frac{\pi+2\theta}{2p}\sin\frac{\pi-2\theta}{2p}\right\}$$

$$\times\left\{\sin\frac{3\pi+2\theta}{2p}\sin\frac{3\pi-2\theta}{2p}\right\}\ldots$$

$$=2^{p-1}\left\{\sin^2\frac{\pi}{2p}-\sin^2\frac{2\theta}{2p}\right\}\left\{\sin^2\frac{2\pi}{2p}-\sin^2\frac{3\theta}{2p}\right\}\ldots \qquad (1)$$

Making θ indefinitely small, we have

$$1 = 2^{p-1}\sin^2\frac{\pi}{2p}\sin^2\frac{3\pi}{2p}\ldots . \qquad (2)$$

Dividing (1) by (2), we have

$$\cos\theta=\left\{1-\frac{\sin^2(2\theta/2p)}{\sin^2(\pi/2p)}\right\}\left\{1-\frac{\sin^2(2\theta/2p)}{\sin^2(3\pi/2p)}\right\}\cdots.$$

Now making p indefinitely large, we get

$$\cos\theta=\left\{1-\frac{4\theta^2}{\pi^2}\right\}\left\{1-\frac{4\theta^2}{3^2\pi^2}\right\}\left\{1-\frac{4\theta^2}{5^2\pi^2}\right\}\ldots\text{ad inf.}$$

$$=\prod_{r=1}^{\infty}\left(1-\frac{4\theta^2}{(2r-1)^2\pi^2}\right). \qquad (3)$$

N.B. This result can also be deduced from result (3) of § 10·3 by proceeding as in § 10·51, or from result (5) of § 10·5 by means of the formula

$$\cos\theta=\frac{\sin 2\theta}{2\sin\theta}.$$

10·7. Sum of powers of the reciprocals of natural numbers.

We can write

$$\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots\text{ad inf.}=\frac{\sin\theta}{\theta}$$

$$=1-\frac{\theta^2}{3!}+\frac{\theta^4}{5!}-\ldots\text{ad inf.}$$

Now taking the logarithms of both sides, we have

$$\log\left(1-\frac{\theta^2}{\pi^2}\right)+\log\left(1-\frac{\theta^2}{2^2\pi^2}\right)+\log\left(1-\frac{\theta^2}{3^2\pi^2}\right)+\ldots$$

$$=\log\left(1-\frac{\theta^2}{3!}+\frac{\theta^4}{5!}-\ldots\right). \qquad (1)$$

But

$$\log\left(1-\frac{\theta^2}{\pi^2}\right)=-\left[\frac{\theta^2}{\pi^2}+\frac{1}{2}\cdot\frac{\theta^4}{\pi^4}+\frac{1}{3}\cdot\frac{\theta^6}{\pi^6}+\cdots\right],$$

$$\log\left(1-\frac{\theta^2}{2^2\pi^2}\right)=-\left[\frac{\theta^2}{2^2\pi^2}+\frac{1}{2}\cdot\frac{\theta^4}{2^4\pi^4}+\frac{1}{3}\cdot\frac{\pi^6}{2^6\theta^6}+\dots\right].$$

Therefore, we have

$$-\frac{\theta^2}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots\right]-\tfrac{1}{2}\frac{\theta^4}{\pi^4}\left[\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\dots\right]$$

$$-\tfrac{1}{3}\frac{\theta^6}{\pi^6}\left[\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}-\dots\right]-\dots$$

$$=\log\left[1-\left(\frac{\theta^2}{6}-\frac{\theta^4}{120}+\dots\right)\right]$$

$$=-\left(\frac{\theta^6}{6}-\frac{\theta^4}{120}+\dots\right)-\tfrac{1}{2}\left(\frac{\theta^2}{6}-\frac{\theta^4}{120}+\dots\right)^2-\dots$$

$$=-\frac{\theta^2}{6}-\frac{\theta^4}{180}+\dots \quad . \quad . \quad . \quad (2)$$

As this equation is true for all values of θ, the coefficients of the various powers of θ on either side are equal. Thus

$$-\frac{1}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots\right]=-\frac{1}{6},$$

$$-\frac{1}{2}\frac{1}{\pi^4}\left[\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\dots\right]=-\frac{1}{180},$$

and so on.

Hence $\quad \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots$, i.e., $\sum_{r=1}^{\infty}\frac{1}{r^2}=\frac{\pi^2}{6}$,

$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\dots$, i.e., $\sum_{r=1}^{\infty}\frac{1}{r^4}=\frac{\pi^4}{90}$;

and so on.

COROLLARY. Similarly, from the relation

$$\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)\ldots=\cos\theta$$
$$=1-\theta^2/2!+\theta^4/4!-\ldots,$$

we can easily deduce the results :

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots, \text{ i.e., } \sum_{r=1}^{\infty}\frac{1}{(2r-1)^2}=\frac{\pi^2}{8},$$
$$\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots, \text{ i.e., } \sum_{r=1}^{\infty}\frac{1}{(2r-1)^4}=\frac{\pi^4}{96},$$

and so on.

NOTE. These results may also be derived from the previous results by following the method used in Ex. 5 solved below.

Ex. 1. Express $\cosh\theta$ as an infinite product.

In the expansion of $\cos\theta$ replacing θ by θi, we have

$$\cos\theta i=\prod_{r=1}^{\infty}\left\{1-\frac{4\theta^2 i^2}{(2r-1)^2\pi^2}\right\}.$$
$$=\prod_{r=1}^{\infty}\left\{1+\frac{4\theta^2}{(2r-1)^2\pi^2}\right\}.$$

But $\cos\theta i=\cosh\theta$.

$$\text{Therefore, } \cosh\theta=\prod_{r=1}\left\{1+\frac{4\theta^2}{(2r-1)^2\pi^2}\right\}.$$

NOTE. Similarly, it can be shown that

$$\sinh\theta=\prod_{r=1}\left(1+\frac{\theta^2}{r^2\pi^2}\right).$$

Ex. 2. Show that

$$\frac{2\,.\,4\,.\,6\ldots 2n}{1\,.\,3\,.\,5\ldots(2n-1)}=\sqrt{(n\pi)},$$

when n is large. [*Travancore*, '51]

When n is large, we can write

$$\sin\theta=\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\ldots\left(1-\frac{\theta^2}{n^2\pi^2}\right) \text{ approx.}$$

whence, putting $\theta=\frac{1}{2}\pi$, we get

$$1=\frac{\pi}{2}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{(2n)^2}\right)$$
$$=\frac{\pi}{2}\cdot\frac{3}{4}\cdot\frac{15}{16}\cdots\frac{(2n)^2-1}{(2n)^2}$$
$$=\frac{\pi}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdots\frac{(2n-1)(2n+1)}{2n\,.\,2n}$$
$$=\frac{\pi}{2}\cdot\frac{1^2\,.\,3^2\,.\,5^2\ldots(2n-1)^2\,.\,(2n+1)}{2^2\,.\,4^2\,.\,6^2\ldots(2n)^2}$$

or
$$\frac{\pi}{2}(2n+1)=\frac{2^2.4^2\ldots(2n)^2}{1^2\,.\,3^2\ldots(2n-1)^2}.$$

Therefore
$$\sqrt{\{\pi(n+\tfrac{1}{2})\}}=\frac{2\,.\,4\,.\,6\ldots 2n}{1\,.\,3\,.\,5\ldots(2n-1)},$$

or, since n is large,
$$\sqrt{(n\pi)}=\frac{2\,.\,4\,.\,6\ldots 2n}{1\,.\,3\,.\,5\ldots(2n-1)}.$$

Note. This is known as Wallis' formula.

Ex. 3. Show that
$$\sin\pi\theta+\cos\pi\theta=(1+4\theta)\prod_{r=1}^{\infty}\left\{1-\frac{4\theta}{4r-1}\right\}\left\{1+\frac{4\theta}{4r+1}\right\}.$$

[*Annam.*, 1939]

We have
$$\sin\pi\theta+\cos\pi\theta=\sqrt{2}\sin(\tfrac{1}{4}\pi+\pi\theta)$$
$$=\sqrt{2}\sin\tfrac{1}{4}\pi(1+4\theta).$$
$$=(\sqrt{2})\tfrac{1}{4}\pi(1+4\theta)\prod_{r=1}^{\infty}\left[1-\frac{\{\frac{1}{4}\pi(1+4\theta)\}^2}{r^2\pi^2}\right]. \quad . \quad . \quad . \quad (1)$$

Putting $\theta=0$, we have
$$1=\tfrac{1}{4}\pi\sqrt{2}\prod_{r=1}^{\infty}\left[1-\frac{1}{(4r)^2}\right]. \quad . \quad . \quad . \quad (2)$$

Dividing (1) by (2), we finally get
$$\sin\pi\theta+\cos\pi\theta=(1+4\theta)\prod_{r=1}^{\infty}\left[\frac{(4r)^2-(1+4\theta)^2}{(4r)^2-1}\right]$$

$$=(1+4\theta)\prod_{r=1}^{\infty}\left[\frac{4r-1-4\theta}{4r-1}\right]\left[\frac{4r+1+4\theta}{4r+1}\right]$$

$$=(1+4\theta)\prod_{r=1}^{\infty}\left\{1-\frac{4\theta}{4r-1}\right\}\left\{1+\frac{4\theta}{4r+1}\right\}.$$

Ex. 4. Prove that

$$\tan x=8x\sum_{r=1}^{\infty}\frac{1}{(2r-1)^2\pi^2-4x^2}. \quad [\textit{Delhi}, 1933]$$

Since $$\cos x=\prod_{r=1}^{\infty}\left[1-\frac{4x^2}{(2r-1)^2\pi^2}\right],$$

therefore, on taking the logarithms of both sides, we get

$$\log\cos x=\sum_{r=1}^{\infty}\log\left\{1-\frac{4x^2}{(2r-1)^2\pi^2}\right\}.$$

Differentiating both sides of this with respect to x, we finally get

$$\tan x=\sum_{r=1}^{\infty}\frac{8x}{(2r-1)^2\pi^2-4x^2}=8x\sum_{r=1}^{\infty}\frac{1}{(2r-1)^2\pi^2-4x^2}.$$

Ex. 5. If $a, b, c, \ldots$ denote all the prime numbers $2, 3, 5, \ldots$ prove that

$$\left(1+\frac{1}{a^2}\right)\left(1+\frac{1}{b^2}\right)\left(1+\frac{1}{c^2}\right)\ldots=15/\pi^2.$$

We know from § 10·7 that

$$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots \qquad (1)$$

$$\therefore \qquad \frac{1}{2^2}\cdot\frac{\pi^2}{6}=\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\ldots . \qquad (2)$$

Therefore, subtracting (2) from (1), we have

$$\frac{\pi^2}{6}\left(1-\frac{1}{2^2}\right)=1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots, \qquad (3)$$

where terms of the type $1/(2r)^2$ are missing.

Multiplying each side of (3) by $1/3^2$

$$\frac{\pi^2}{6}\left(1-\frac{1}{2^2}\right)\cdot\frac{1}{3^2}=\frac{1}{3^2}+\frac{1}{9^2}+\dots \quad . \quad . \quad . \quad (4)$$

Therefore, subtracting (4) from (3), we have

$$\frac{\pi^2}{6}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)=1+\frac{1}{5^2}+\frac{1}{7^2}+\dots,$$

where terms of the type $(1/3r)^2$ also are absent.

Proceeding in this way, we finally get

$$\frac{\pi^2}{6}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\dots=1, \quad . \quad . \quad . \quad (5)$$

where in the brackets on the left the denominators 2, 3, 5, ... are all the prime numbers.

Similarly, from

$$\frac{\pi^4}{90}=1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\dots$$

we get

$$\frac{\pi^4}{90}\left(1-\frac{1}{2^4}\right)\left(1-\frac{1}{3^4}\right)\left(1-\frac{1}{5^4}\right)\dots=1. \quad . \quad . \quad . \quad (6)$$

Hence, dividing (6) by (5), we obtain

$$\frac{\pi^2}{15}\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{5^2}\right)\dots=1$$

or

$$\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\left(1+\frac{1}{5^2}\right)\dots=\frac{15}{\pi^2},$$

or, in the terminology of the question,

$$\left(1+\frac{1}{a^2}\right)\left(1+\frac{1}{b^2}\right)\left(1+\frac{1}{c^2}\right)\dots=\frac{15}{\pi^2}.$$

Examples

1. Prove that

(i) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\dots$ ad inf. $=\pi^2/12$. [*Cal.*, '51]

(ii) $\frac{1}{1.2}+\frac{1}{2.4}+\frac{1}{3.6}+\frac{1}{4.8}+\dots$ ad inf. $=\pi^2/12$.

(iii) $\left\{\dfrac{1}{1.2.3}\right\}^2+\left\{\dfrac{1}{2.3.4}\right\}^2+\left\{\dfrac{1}{3.4.5}\right\}^2$

$$+\ldots\text{ad inf.}=\frac{\pi^2}{4}-\frac{39}{16}.$$

2. Prove that the sum of the products taken two at a time of the squares of the reciprocals of

(i) all positive integers is $\pi^4/120$,

(ii) all positive odd integers is $\pi^4/384$. [*Utkal*, 1954]

3. Show that

$$\text{(i)}\ \sum_1^\infty \frac{1}{n^2(n+1)^2}=\frac{\pi^2}{3}-3.$$

$$\text{(ii)}\ \sum_1^\infty \frac{1}{n^3(n+1)^3}=10-\pi^2.$$

4. Show that

$$\tfrac{1}{2}\pi=\frac{2.2.4.4.6.6.8.8\ldots}{1.3.3.5.5.7.7.9\ldots}.$$ [*Cal.*, '50]

5. Prove that

$$\text{(i)}\quad \pi=3.\frac{36}{35}.\frac{144}{143}.\frac{324}{323}.\frac{576}{575}\cdots.$$

$$\text{(ii)}\ \sqrt{2}=\frac{4.36.100.196.324\ldots}{3.35.\ 99.195.323\ldots}.$$

$$\text{(iii)}\ \frac{\sqrt{3}}{2}=\frac{8.80.224.440\ldots}{9.81.225.441\ldots}.$$

6. Deduce the expression for $\sin x$ in factors from that for $\cos x$.

7. Prove that

$$(1+\sin x)=\tfrac{1}{8}(\pi+2x)^2\left\{1-\frac{(\pi+2x)^2}{4^2\pi^2}\right\}^2 \times\left\{1-\frac{(\pi+2x)^2}{8^2\pi^2}\right\}^2\cdots.$$

8. Prove that

$$\cos(\tfrac{1}{2}\pi\sin\theta)=\tfrac{1}{4}\pi\cos^2\theta\left(1+\frac{\cos^2\theta}{2.4}\right)\left(1+\frac{\cos^3\theta}{4.6}\right)\cdots.$$

9. Prove that $\cos x + \sin x$

$$= \frac{\pi+4x}{2\sqrt{2}}\left\{1-\frac{(\pi+4x)^2}{4^2\pi^2}\right\}\left\{1-\frac{(\pi+4x)^2}{8^2\pi^2}\right\}\dots .$$

10. Show that

$$\cos a + \tan x \sin a = \left(1+\frac{2a}{\pi-2x}\right)\left(1-\frac{2a}{\pi+2x}\right)$$
$$\times\left(1+\frac{2a}{3\pi-2x}\right)\left(1-\frac{2a}{3\pi+2x}\right)\dots ;$$

Hence, or otherwise, prove that

$$\tan x = \frac{2}{\pi-2x} - \frac{2}{\pi+2x} + \frac{2}{3\pi-2x} - \frac{2}{3\pi+2x} + \dots$$
$$= 8x\sum_{1}^{\infty}\frac{1}{(2n-1)^2\pi^2-4x^2}.$$

[*Cal.*, 1931]

11. Prove that

$$\cos a - \cot x \sin a = \left(1-\frac{a}{x}\right)\left(1+\frac{a}{\pi-x}\right)\left(1-\frac{a}{\pi+x}\right)$$
$$\times\left(1+\frac{a}{2\pi-x}\right)\left(1-\frac{a}{2\pi+x}\right)\dots .$$

Hence, or otherwise, show that

$$\cot x = \frac{1}{x} - \frac{1}{\pi-x} + \frac{1}{\pi+x} - \frac{1}{2\pi-x} + \frac{1}{2\pi+x} - \dots$$
$$= \frac{1}{x} + 2x\sum_{1}^{\infty}\frac{1}{x^2-r^2\pi^2}.$$

[*Cal.*, 1947]

12. Show that

$$\sin x + \cos x = \left(1+\frac{4x}{\pi}\right)\left(1-\frac{4x}{3\pi}\right)\left(1+\frac{4x}{5\pi}\right)\left(1-\frac{4x}{7\pi}\right)\dots,$$

and hence deduce that

$$1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\dots = \frac{\pi^2}{32}.$$

[*Nagpur*, 1940]

13. Show that

(i) $\dfrac{\pi}{3\sqrt{3}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \frac{1}{10} - \dots$

(ii) $\dfrac{\pi}{2\sqrt{3}}=1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\frac{1}{13}-\frac{1}{17}+\frac{1}{19}-\ldots$

[*Cal.*, 1935]

14. Using the relation $\operatorname{cosec}\theta=\frac{1}{2}\left(\tan\dfrac{\theta}{2}+\cot\dfrac{\theta}{2}\right)$,

prove that

$$\operatorname{cosec}\theta=\frac{1}{\theta}-\frac{1}{\theta-\pi}-\frac{1}{\theta+\pi}+\frac{1}{\theta-2\pi}+\frac{1}{\theta+2\pi}-\frac{1}{\theta-3\pi}-\frac{1}{\theta+3\pi}+\ldots$$

$$=\frac{1}{\theta}+2\theta\sum_{n=1}^{\infty}\frac{(-1)^n}{\theta^2-n^2\pi^2}.$$

[*Cal.*, 1948]

15. Prove that

$$\frac{1}{4\pi}\sec x=\frac{1}{\pi^2-4x^2}-\frac{3}{3^2\pi^2-4x^2}+\frac{5}{5^2\pi^2-4^2x^2}-\ldots.$$

[*Cal.*, 1936]

16. Prove that

$$\frac{1}{\sin^2 x}=\sum_{-\infty}^{\infty}\frac{1}{(x+n\pi)}.$$

[*Annam.*, 1943]

17. Show that

$$\frac{1}{4\cos^2\theta}=\frac{1}{(\pi-2\theta)^2}+\frac{1}{(\pi+2\theta)^2}+\frac{1}{(3\pi-2\theta)^2}+\frac{1}{(3\pi+2\theta)^2}+\ldots.$$

18. Prove that

$$\frac{\sin(\alpha+\theta)}{\sin\alpha}=\Pi\left(1+\frac{\theta}{\alpha+r\pi}\right),$$

where r is any positive or negative integer, including zero.

[*Bombay*, 1947]

19. Prove that

$$\frac{\cos\theta-\cos\alpha}{1-\cos\alpha}=\Pi\left\{1-\frac{\theta^2}{(\alpha+r\pi)^2}\right\},$$

where r is any even positive or negative integer, including zero. Hence deduce the factors of $\cosh x-\cos\alpha$.

20. Show that

$$\frac{\sin\alpha-\sin x}{\sin\alpha}=\left(1-\frac{x}{\alpha}\right)\left(1+\frac{x}{\pi-\alpha}\right)\left(1-\frac{x}{\pi+\alpha}\right)$$
$$\times\left(1+\frac{x}{2\pi-\alpha}\right)\left(1-\frac{x}{2\pi+\alpha}\right)\cdots.$$

21. Find the value of the infinite product

$$\left(1+\frac{1}{1^2}\right)\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{3^2}\right)\cdots.$$ [Delhi, 1931]

22. Prove that

$$\cosh(\cos\theta)=\frac{e^2+1}{2e}\prod_{n=1}^{\infty}\left\{1-\frac{4\sin^2\theta}{(2n-1)^2\pi^2+4}\right\}.$$

[Annam., 1947]

23. Prove that

$$\cosh\theta+\cos\alpha=2\cos^2\tfrac{1}{2}\alpha\,\Pi\left\{1+\frac{\theta^2}{(\alpha+r\pi)^2}\right\},$$

where r is any odd positive or negative integer. [Delhi, 1950]

24. Establish the equality

$$\prod_{r=1}^{\infty}\frac{2^2r^2-1}{2^2r^2+1}=\operatorname{cosech}\frac{\pi}{2}.$$ [Cal., 1948]

25. Prove that

$$\left\{1+\frac{2}{1+1^2}+\frac{2}{1+2^2}+\frac{2}{1+3^2}+\cdots\right\}$$
$$\times\left\{\frac{1}{4+1^2}+\frac{1}{4+3^2}+\frac{1}{4+5^2}+\cdots\right\}=\frac{\pi^2}{8}.$$

[Delhi, 1933]

26. By writing $\sin(x+iy)$ as a product or otherwise, prove that

$$\tan^{-1}(\tanh y\cot x)=\tan^{-1}\frac{y}{x}-\sum_{1}^{\infty}\tan^{-1}\frac{2xy}{n^2\pi^2-x^2+y^2},$$

and hence deduce that

$$\tan^{-1}\frac{1}{\pi^2}+\tan^{-1}\frac{1}{4\pi^2}+\tan^{-1}\frac{1}{9\pi^2}+\tan^{-1}\frac{1}{16\pi^2}+\cdots$$
$$=\tfrac{1}{4}\pi-\tan^{-1}\left(\tanh\frac{1}{\sqrt{2}}\cot\frac{1}{\sqrt{2}}\right).$$

27. Show that

$$\frac{\sin\theta}{\theta}=1-\frac{1}{\pi^2}\cdot\frac{\theta^2}{1^2}+\frac{1}{\pi^4}\cdot\frac{\theta^2(\theta^2-\pi^2)}{1^2 \cdot 2^2}$$
$$-\frac{1}{\pi^6}\cdot\frac{\theta^2(\theta^2-\pi^2)(\theta^2-2^2\pi^2)}{1^2 \cdot 2^2 \cdot 3^2}+\dots$$

and deduce that

$$\frac{1}{1^2}\cdot\frac{1}{2^2}+\left(\frac{1}{1^2}+\frac{1}{2^2}\right)\frac{1}{3^2}+\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}\right)\frac{1}{4^2}+\dots=\frac{1}{120}.$$

[*Cal.*, 1933]

28. Show that

$$\cos\left(\tfrac{1}{2}\pi\theta\right)=1-\theta^2-(1-\theta^2)\frac{\theta^2}{9}-(1-\theta^2)\left(1-\frac{\theta^2}{9}\right)\frac{\theta^2}{25}-\dots,$$

and deduce that

$$\frac{1}{3^2}+\left(1+\frac{1}{3^2}\right)\frac{1}{5^2}+\left(1+\frac{1}{3^2}+\frac{1}{5^2}\right)\frac{1}{7^2}+\dots=\frac{\pi^4}{384}.$$

29. If 2, 3, 5, ... are all prime numbers, show that

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{5^2}\right)\dots=\frac{6}{\pi^2}.$$

[*Delhi*, 1953]

30. If 2, 3, 5, ... are all prime numbers, show that

$$\frac{2^4}{2^4+1}\cdot\frac{3^4}{3^4+1}\cdot\frac{5^4}{5^4+1}\dots=\frac{\pi^4}{105}$$

MISCELLANEOUS EXAMPLES

1. Show that

$$\tan\left(\tfrac{1}{2}\sin^{-1}\frac{2x}{1+x^2}+\tfrac{1}{2}\cos^{-1}\frac{1-x^2}{1+x^2}\right)=\frac{2x}{1-x^2}.$$ [*Gauhati*, '51]

2. Prove that

$$\tan(\tan^{-1}x+\tan^{-1}y+\tan^{-1}z)=\cot(\cot^{-1}x+\cot^{-1}y+\cot^{-1}z).$$

[*Nagpur*, '35]

3. Show that

$$\tan^{-1}\frac{a(b-c)}{1+a^2bc}+\tan^{-1}\frac{b(c-a)}{1+b^2ca}+\tan^{-1}\frac{c(a-b)}{1+c^2ba}=n\pi,$$

n being an integer or zero. [*Andhra*, 1936]

4. Prove that

$$\sin^{-1}x+\sin^{-1}y=\sin^{-1}(x\sqrt{1-y^2}+y\sqrt{1-x^2}).$$

Explain the fallacy in the following :

Putting $x=y=\sqrt{3}/2$ in the above formula, we get

$$\pi/3+\pi/3=\sin^{-1}\sqrt{3}/2=\pi/3.$$

$$\therefore\quad 120^\circ=60^\circ.$$

5. If a and b be two complex numbers, show that the modulus of $a-b$ is the distance between the points which represent a and b in the complex plane. [*Travancore*, '47]

6. If z_1, z_2, z_3 are the vertices of an equilateral triangle drawn in the Argand diagram, prove that

$$z_1^2+z_2^2+z_3^2=z_2z_3+z_3z_1+z_1z_2.$$ [*I.C.S.*, 1943]

7. Prove that if the amplitude of $(z_1-z_3)(z_2-z_4)/(z_1-z_4)(z_2-z_3)$ is zero, the four points representing z_1, z_2, z_3, z_4 in the Argand diagram lie on a circle or a straight line.

[*Panjab*, 1945]

8. Show that

$$\{\cos\theta+\cos\phi+i(\sin\theta+\sin\phi)\}^n+\{\cos\theta+\cos\phi-i(\sin\theta+\sin\phi)\}^n$$
$$=2^{n+1}\{\cos\tfrac{1}{2}(\theta-\phi)\}^n\,.\,\cos\tfrac{1}{2}n(\theta+\phi).$$

[*Calcutta*, 1949]

9. Express $\{2+3\sqrt{(-1)}\}^7$ in the form $A+iB$.

10. If $x=\cos\theta+i\sin\theta$, prove that

$$\frac{1}{2x+}\,\frac{1}{2x+}\,\frac{1}{2x+}\ldots\text{ad inf.}=[(\cos\theta+\cos^2\theta)^{1/2}-\cos\theta]$$
$$+i[(\cos\theta-\cos^2\theta)^{1/2}-\sin\theta].$$

[*Lucknow*, '65]

11. Prove that the roots of the equation

$$x^{10}+11x^5-1=0$$

are the values of

$$\tfrac{1}{2}(\pm\sqrt{5}-1)(\cos\tfrac{2}{5}r\pi\pm i\sin\tfrac{2}{5}r\pi).$$

12. Find the complex numbers z which satisfy the equation

$$(z+1)^5=(z-i)^5,$$

and show that when they are marked in the complex plane they lie on a straight line. [*Travancore*, '47]

13. Solve $(\cot\theta+i)^5+(\cot\theta-i)^5=0$,
and hence find the value of $\cot\pi/10$. [*Travancore*, '42]

14. If α, β be the roots of $t^2-2t+2=0$, show that

$$\frac{(x+\alpha)^n-(x+\beta)^n}{\alpha-\beta}=\frac{\sin n\phi}{\sin^n\phi},$$

where $\phi=\cot^{-1}(x+1)$. [*Lucknow*, '58]

15. If ω be a complex cube root of unity, prove that

$$1+\omega^n+\omega^{2n}$$

is equal to 3 or 0, according as n is or is not a multiple of 3.

16. By putting $a=\text{cis}\,2\alpha$, $b=\text{cis}\,2\beta$ and $c=\text{cis}\,2\gamma$, in the identity

$$a(b-c)+b(c-a)+c(a-b)=0$$

prove that

$$\Sigma\cos(2\alpha+\beta+\gamma)\sin(\beta-\gamma)=0.$$ [*Travancore*, '47]

17. If the product P of the n binomials $\cos\alpha_1+i\sin\alpha_1$, $\cos\alpha_2+i\sin\alpha_2$, ..., $\cos\alpha_n+i\sin\alpha_n$ be an imaginary cube root of unity, prove that

$$3(\alpha_1+\alpha_2+\ldots+\alpha_n)/2\pi$$

must be an integer of the form $3k+1$, or $3k+2$, k being an integer, positive or negative.

18. From the identity

$$\frac{x^{m+1}-y^{m+1}}{x-y}=x^m+x^{m-1}y+x^{m-2}y^2+\dots+y^m$$

deduce the sum of all the values of $\cos(r\alpha+s\beta)$ where r and s take all possible integral values including zero such that $r+s=m$. [*Banaras*, 1936]

19. Show that $2\cos\frac{2\pi}{7}$ is a root of

$$x^3+x^2-2x-1=0. \qquad [\textit{Travancore}, \text{'44}]$$

20. Show that $\sin\frac{2\pi}{7}$, $\sin\frac{4\pi}{7}$, $\sin\frac{8\pi}{7}$ are the roots of the equation

$$x^3-\frac{\sqrt{7}}{2}x^2+\frac{\sqrt{7}}{8}=0. \qquad [\textit{Delhi}, 1948]$$

21. Show that $\sin\frac{\pi}{14}$ is a root of the equation

$$64x^6-80x^4+24x^2-1=0. \qquad [\textit{Lucknow}, \text{'56}]$$

22. If $\alpha=\cos\frac{2\pi}{13}+\cos\frac{6\pi}{13}+\cos\frac{18\pi}{13}$

and $\beta=\cos\frac{10\pi}{13}+\cos\frac{14\pi}{13}+\cos\frac{22\pi}{13}$,

show that they are the roots of

$$4x^2+2x-3=0.$$

Evaluate α, β. [*Travancore*, 1947]

23. Given that $V=\dfrac{a\sin\theta+b\sin 2\theta+\sin 3\theta}{\theta^5}$

tends to a finite limit, when the angle θ tends to zero, determine the constants a and b, and the consequent value of V. [*Andhra*, 1937]

24. Show that $\dfrac{3\sin\theta}{2+\cos\theta}$ differs from θ by $\theta^5/180$, where θ is small, positive and in radians. [*Travancore*, 1947]

25. If $x \cot x = a_0 + a_2 x^2 + a_4 x^4 + \ldots + a_{2n} x^{2n} + \ldots$, shew that

$$a_{2n} = \frac{a_{2n-2}}{3!} - \frac{a_{2n-4}}{5!} + \ldots + \frac{(-1)^{n-1} a_0}{(2n+1)!} + \frac{(-1)^n}{(2n)!}.$$

Also show that the first four terms in the expansion of $x \cot x$ in ascending powers of x are

$$1 - \frac{x^2}{3} - \frac{x^4}{45} - \frac{2x^6}{945}.$$

26. Prove that there are five values of θ not differing from one another by a multiple of π which satisfy the equation

$$a \tan 3\theta + b \tan 2\theta + c \tan \theta = d,$$

and if these be $\alpha, \beta, \gamma, \delta, \epsilon$, show that

$$\alpha + \beta + \gamma + \delta + \epsilon = n\pi + \tan^{-1} \frac{d}{a+b+c}. \quad [\textit{Nagpur}, 1931]$$

27. Show that

$$\lim_{\beta \to \alpha} \frac{\alpha \sin \beta - \beta \sin \alpha}{\alpha \cos \beta - \beta \cos \alpha} = \tan (\alpha - \tan^{-1} \alpha). \quad [\textit{Travancore}, \text{'51}]$$

28. If $x = \log \tan \left(\frac{\pi}{4} + \frac{1}{2} y\right)$, prove that

$$y = -i \log \tan (\tfrac{1}{4}\pi + \tfrac{1}{2} xi).$$

29. If $x^{x^{x \cdots \text{ad inf}}} = a(\cos \theta + i \sin \theta)$, show that the general value of x is

$$r(\cos \phi + i \sin \phi)$$

where
$$a \log r = (2n\pi + \theta) \sin \theta + \log a \,.\, \cos \theta$$
$$a\phi = (2n\pi + \theta) \cos \theta - \log a \,.\, \sin \theta.$$

30. Express $4 \sinh 2x \sinh 4x \sinh 6x$ as a sum.

31. If $(1 + e \cosh \theta)(1 - e \cosh \phi) = 1 - e^2$, show that

$$\frac{\tanh \frac{1}{2}\theta}{\tanh \frac{1}{2}\phi} = \sqrt{\left(\frac{1+e}{1-e}\right)}.$$

32. Show that

$$\tan^{-1} \frac{\tan 2\theta + \tanh 2\phi}{\tan 2\theta - \tanh 2\phi} + \tan^{-1} \frac{\tan \theta - \tanh \theta}{\tan \theta + \tanh \theta}$$
$$= \tan^{-1} (\cot \theta \coth \phi).$$

33. If $\sin(\theta+i\phi)=\cos\alpha+i\sin\alpha$, prove that

$$\cos^2\theta=\pm\sin\alpha.$$

34. If $\sin(\theta+i\phi)=A(\cos\alpha+i\sin\alpha)$, show that

(i) $2A^2=\cosh 2\phi-\cos 2\theta$,

(ii) $\tanh\phi=\tan\alpha\tan\theta$.

35. If $\cosh q=\sec p$, prove that

$$e^{ip}=\text{sech}\, q+i\tanh q.$$

36. If $\log\log\log(x+iy)=p+iq$, show that

(i) $\exp(e^p\cos q)\cos(e^p\sin q)=\frac{1}{2}\log(x^2+y^2)$.

(ii) $\exp(e^p\cos q)\sin(e^p\sin q)=\tan^{-1}(y/x)$.

37. If $u+iv=\text{cosec}\,(x+iy)$, show that

$$(u^2+v^2)^2=\frac{u^2}{\sin^2 x}-\frac{v^2}{\cos^2 x}.$$ [*Lucknow*, '65]

38. If $X+iY=\cosh^{-1}(x+iy)$, prove that $X=$const. and $Y=$const. represent a family of ellipses and hyperbolas respectively. [*I. C. S.*, 1931]

39. If $\cosh u=\sec\theta$, show that

$$\sinh u=\tan\theta,\ \tanh u=\sin\theta,$$

$$\tan^2\tfrac{1}{2}\theta=\tanh^2\tfrac{1}{2}u \text{ and } u=\log_e\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\theta).$$

40. Show that, if α, β, γ are the cube roots of unity,

$$\frac{\tan^{-1}\alpha}{\alpha}+\frac{\tan^{-1}\beta}{\beta}+\frac{\tan^{-1}\gamma}{\gamma}=\frac{\pi}{2}+\frac{\sqrt{3}}{4}\log\frac{2+\sqrt{3}}{2-\sqrt{3}}$$
$$=3(1-\tfrac{1}{7}+\tfrac{1}{13}-\tfrac{1}{19}+\tfrac{1}{25}-\dots).$$

41. From the expansion of $\sin^n\theta$ in terms of sines of multiples of θ, deduce that

$$n^p-\frac{n}{1}(n-2)^p+\frac{n(n-1)}{1.2}(n-4)^p+\dots\text{to }\frac{n+1}{2}\text{terms}=0,$$

where n and p are odd positive integers and $p<n$.

Find the value of the series when $p=n$.

[*Utkal*, 1954]

42. Prove that the determinant of the nth order

$$\begin{vmatrix} 2\cos\theta & -1 & 0 & . & 0 & 0 \\ -1 & 2\cos\theta & -1 & . & 0 & 0 \\ 0 & -1 & 2\cos\theta & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & -1 & 2\cos\theta \end{vmatrix}$$

is equal to $\dfrac{\sin(n+1)\theta}{\sin\theta}$.

43. Expand $\dfrac{\cos\theta-x}{1-2x\cos\theta+x^2}$ in a series of ascending powers of x, when $|x|$ is less than unity.

44. If $\tan(\theta+\phi)\cos 2\alpha=\tan\phi$, prove that

$$\theta=\tan^2\alpha\sin 2\phi+\tfrac{1}{2}\tan^4\alpha\sin 4\phi+\tfrac{1}{3}\tan^6\alpha\sin 6\phi+\dots.$$

45. If the roots of the equation $ax^2+bx+c=0$ be imaginary, show that the coefficient of x^n in the development of $(ax^2+bx+c)^{-1}$ in powers of x is

$$a^{n/2}\sin(n+1)\theta\div c^{n/2+1}\sin\theta,$$

where θ is given by $b\sec\theta+2\sqrt{(ac)}=0$. [*Utkal*, 1950]

46. Prove that, if $p<1$, the infinite series

$$1+np\cos\theta+\frac{n(n+1)}{2}p^2\cos 2\theta+\frac{n(n+1)(n+2)}{3}p^3\cos 3\theta+\dots$$
$$=q^n\cos n\phi,$$

where $1/q^2=1-2p\cos\theta+p^2$ and $\sin\phi=pq\sin\theta$.

[*Rangoon*, 1938]

47. If n is odd, prove that $S=3C=n^2-1$, where

$$S=\sec^2\frac{\pi}{n}+\sec^2\frac{2\pi}{n}+\dots+\sec^2\frac{(n-1)\pi}{n},$$

and $$C=\operatorname{cosec}^2\frac{\pi}{n}+\operatorname{cosec}^2\frac{2\pi}{n}+\dots+\operatorname{cosec}^2\frac{(n-1)\pi}{n}.$$

[*Travancore*, 1951]

48. Show that

$$1+2\cos 2x+2\cos 4x+\dots+2\cos 2nx=\frac{\sin(2n+1)x}{\sin x}.$$

[*Annam.*, 1936]

49. Prove that

$$\cosh x \cosh y = 1 + \frac{r^2 \cosh 2\alpha}{2!} + \frac{r^4 \cosh 4\alpha}{4!} + \dots + \frac{r^{2n} \cosh 2n\alpha}{2n!} + \dots$$

where $\tanh \alpha = y/x$ and $r^2 = x^2 - y^2$.

50. Prove that the sum of the infinite series

$$1 + \frac{\cos 4\theta}{4!} + \frac{\cos 8\theta}{8!} + \frac{\cos 12\theta}{12!} + \dots$$

is $\frac{1}{2}\{\cos(\cos\theta)\cosh(\sin\theta) + \cos(\sin\theta)\cosh(\cos\theta)\}$.

51. Sum to infinity

$$\frac{5\cos\theta}{1} + \frac{7\cos 3\theta}{3!} + \frac{9\cos 5\theta}{5!} + \dots$$ [*Banaras*, '51]

52. If s_n denotes the sum of n terms of the series

$$1 + 2\cos\theta + 2\cos 2\theta + 2\cos 3\theta + \dots,$$

show that

$$s_1 + s_2 + \dots + s_n = \sin^2 \tfrac{1}{2}n\theta \operatorname{cosec}^2 \tfrac{1}{2}\theta.$$

[*Lucknow*, 1954]

53. Show that the sum of the series

$$\cos\alpha + x\cos(\alpha+\beta) + x^2\cos(\alpha+2\beta) + \dots \text{to } n \text{ terms},$$

where $n\beta = 2\pi$ and x is any one of the nth roots of unity, vanishes with two exceptions; and find the sum in these two cases.

[*Utkal*, 1951]

54. Find the sum of $(n+1)$ terms of the series

$$\cos\theta + nx\cos(\theta-\phi) + \frac{n(n-1)}{2!}x^2\cos(\theta-2\phi) + \dots .$$

55. If $u = \sum_{n=0}^{\infty} (-1)^n \frac{\sin(2n+1)\theta \sin^{2n+1}\alpha}{2n+1}$

show that

$$\sin\theta = \tfrac{1}{2}(\operatorname{cosec}\alpha + \sin\alpha)\tanh 2u.$$

56. Prove that

$$\cos 2n\theta = \left(1 - \frac{\sin^2\theta}{\sin^2\pi/4n}\right)\left(1 - \frac{\sin^2\theta}{\sin^2 3\pi/4n}\right)\dots .$$

What is the last factor in this product ?

Deduce, or otherwise prove, that

$$\cos \tfrac{1}{2}\theta = \left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\left(1-\frac{\theta^2}{5^2\pi^2}\right)\cdots.$$

57. Show that

$$\sum_{r=0}^{n-1} \cot^2 \frac{4r+1}{4n}\pi = n(2n-1).$$

58. Prove that, when n is even,

$$\cos n\theta = \cos^n \theta \prod_{r=0}^{n/2-1}\left[1-\tan^2\theta/\tan^2\frac{2r+1}{2n}\pi\right],$$

and deduce the infinite product formula for $\cos\theta$.

59. Show that the value of the infinite product

$$\left(1-\tan^2\frac{\theta}{2}\right)\left(1-\tan^2\frac{\theta}{2^2}\right)\left(1-\tan^2\frac{\theta}{2^3}\right)\cdots$$

is $\theta/\tan\theta$. [*Nagpur*, 1940]

60. Assuming the formula for $\sin\theta$ as an infinite product, prove that

$$\left(1+\frac{x}{1}\right)\left(1-\frac{x}{5}\right)\left(1+\frac{x}{7}\right)\left(1-\frac{x}{11}\right)\cdots = \cos\frac{\pi x}{6}+\sqrt{3}\sin\frac{\pi x}{6},$$

where the signs alternate in the factors, and the denominators are odd integers not divisible by 3 in ascending order.

Hence sum the infinite series

$$1-\frac{1}{5^3}+\frac{1}{7^3}-\frac{1}{11^3}+\frac{1}{13^3}-\frac{1}{17^3}+\frac{1}{19^3}-\frac{1}{23^3}+\cdots.$$

[*Bombay*, 1947]

61. Prove that

$$\prod_{n=2}^{\infty}\left(1-\frac{1}{n^4}\right) = \frac{e^{\pi}-e^{-\pi}}{8\pi}.$$ [*Travancore*, 1942]

62. Prove that the sum of the squares of the reciprocals of all numbers which are not divisible by the square of any prime number is $15/\pi^2$. [*Banaras*, 1935]

63. Prove that

(i) $1+\sin\theta=\left(1+\frac{2\theta}{\pi}\right)^2\left(1-\frac{2\theta}{3\pi}\right)^2\left(1+\frac{2\theta}{5\pi}\right)^2\cdots.$

(ii) $\tan\frac{\pi(1+x)}{4}=\frac{1+x}{1-x}\cdot\frac{3-x}{3+x}\cdot\frac{5+x}{5-x}\cdot\frac{7-x}{7+x}\cdots.$

64. Show that the sum of the infinite series

$$\frac{1}{1^2+x^2}+\frac{1}{3^3+x^2}+\frac{1}{5^2+x^2}+\ldots$$

is $(\pi/4x)\tanh(\pi x/2)$. [*Banaras*, 1947]

65. Sum the series

$\sin\alpha+x\sin(\alpha+\beta)+x^2\sin(\alpha+2\beta)+\ldots$ to n terms. [*Lucknow*, '67]

ANSWERS

PAGES 7-10

9. $x=\pm 1/\sqrt{2}$. **10.** $x=-461/9$. **11.** $x=2$.

12. 1. **13.** 1/2257. **16.** $x=0, \pm 1/2$.

17. $x=(a-b)/(1+ab)$. **18.** $x=\pm 1$ or $\pm(1\pm\sqrt{2})$.

19. $x=\pm 1$.

21. $x=\cos[\{(a+b)\pi/2-(\alpha-\beta)\}/2a]$,
$y=\cos[\{(\alpha+\beta)-(a-b)\pi/2\}/2b]$.

25. $\tan\theta=0, \pm 1, -2$. **26.** $x=1, y=2; x=2, y=7$.

27. $n\pi+(-1)^{m+n}.\pi/6$. [The particular value is $(-1)^m.\pi/6$.]

PAGES 20-22

2. (i) $2i$, (ii) $1/2-(1/2)i$, (iii) i, (iv) -1,
(v) $\{8ab(a^2-b^2)/(a^2+b^2)^2\}i$.

3. (i) 1; $\pi/3$ (ii) $2\cos(\theta/2)$; $\theta/2$, (iii) $2\sin(\theta/2)$; $(\pi-\theta)/2$. (iv) $\sqrt{2}; -\pi/4$. (v) 13; θ, such that $\cos\theta=-5/13$ and $\sin\theta=-12/13$. (vi) $\sqrt{13}$; θ, such that $\cos\theta=2/\sqrt{13}$ and $\sin\theta=-3/\sqrt{13}$.

9. If ABC be the triangle whose vertices A, B, C are represented by the complex numbers z_1, z_2, z_3, then mod $\{(z_2-z_1)/(z_3-z_1)\}$ denotes the ratio AB/AC and arg $\{(z_2-z_1)/(z_3-z_1)\}$ denotes the angle BAC.

PAGES 28-30

1. $\cos 2\theta+i\sin 2\theta$. **2.** $\cos 7\theta+i\sin 7\theta$.

3. $\cos 13\theta-i\sin 13\theta$.

4. $-1+i$, $\sqrt{2}[\cos(3\pi/4)+i\sin(3\pi/4)]$.

8. $x^2-2x\cos n\theta+1=0$.

PAGES 34-35

1. (i) $1, -\frac{1}{2}(1-i\sqrt{3}), -\frac{1}{2}(1+i\sqrt{3})$.
(ii) $-1, \frac{1}{2}(1+i\sqrt{3}), \frac{1}{2}(1-i\sqrt{3})$.

2. (i) $2^{1/14}\{\cos \frac{1}{7}(2n\pi+\pi/4)+i\sin\frac{1}{7}(2n\pi+\pi/4)\}$, where $n=$ 0, 1, 2, 3, 4, 5, 6. (ii) $\sqrt[3]{2}(\pm\frac{1}{2}\sqrt{3}+\frac{1}{2}i)$, $-\sqrt[3]{2}.i$.
(iii) $\sqrt[3]{2}\,\{\cos(n\pi/18)+i\sin(n\pi/18)\}$, where $n=1, 13, 25$.
(iv) $2^{1/12}\,[\cos\{(8n+3)\pi/24\}+i\sin\{(8n+3)\pi/24\}]$, where $n=0, 1, 2, 3, 4, 5$.

3. $\cos(5\pi/12)+i\sin(5\pi/12)$; $\pm\{\cos(r\pi/48)+i\sin(r\pi/48)\}$, where $r=5, 29$.

4. $x=1$, $\cos(r\pi/7)\pm i\sin(r\pi/7)$, where $r=2, 4, 6$.

5. $x=1, -1, -1, \pm i$, $\cos(\pi/5)\pm i\sin(\pi/5)$, $\cos(3\pi/5)\pm i\sin(3\pi/5)$.

6. 1. **7.** $-2i$, $(\pm\sqrt{3}+i)$.

8. $\{2\sin(\phi/2)\}^{1/3}.[\cos\{(4n\pi+\pi-\phi)/6\}-i\sin\{(4n\pi+\pi-\phi)/6\}]$ where $n=0, 1, 2$.

9. -2, $(1\pm i\sqrt{3})$.

10. ± 1, $\pm i$, $\frac{1}{2}(\sqrt{3}\pm i)$, $\frac{1}{2}(1\pm i\sqrt{3})$, $\frac{1}{2}(-1\pm i\sqrt{3})$, and $\frac{1}{2}(-\sqrt{3}\pm i)$.

The common roots are $\frac{1}{2}(1\pm i\sqrt{3})$ and $\frac{1}{2}(-1\pm i\sqrt{3})$.

Pages 37-40

1. $x=2\sqrt{2}\{\cos(\pi/4)-i\sin(\pi/4)\}$,
$y=2\,\{\cos(\pi/6)-i\sin(\pi/6)\}$; $x^2y^2=-64$; 1.

6. $\sqrt[3]{2}[\cos\{(6n+1)\pi/9\}-i\sin\{(6n+1)\pi/9\}]$, where $n=0, 1, 2$.

12. (i) $\cos(\pi/5)\pm i\sin(\pi/5)$, $\cos(3\pi/5)\pm i\sin(3\pi/5)$.
(ii) -1, $(1\pm i\sqrt{3})/2$, $\pm(1\pm i)/\sqrt{2}$.
(iii) $\cos(2n\pi/7)+i\sin(2n\pi/7)$, where $n=1, 2, 3, 4, 5, 6$.

13. $2.\,10^{1/6}\cos\frac{1}{3}(2n\pi+\theta)$, where $n=0, 1, 2$, and $\tan\theta=3$.

14. 1, $\cos(r\pi/7)\pm i\sin(r\pi/7)$, where $r=2, 4, 6$.

15. $[\{4-5\cos(2n\pi/5)\}-3i\sin(2n\pi/5)]/\{5-4\cos(2n\pi/5)\}$, where $n=0, 1, 2, 3, 4$.

17. (i) $x=2^{1/4}\{\cos(n\pi/12)-i\sin(n\pi/12)\}$, where $n=1, 7, 13, 19$,
(ii) $x=(1/2)^{1/4}\{\cos(n\pi/24+i\sin(n\pi/24)\}$, where $n=1, 13, 25, 37$.

18. $\theta=2n\pi/r^2$.

PAGES 48-51

7. The particular case is $q=1$.

8. $0, \frac{1}{5}\pi, \frac{2}{5}\pi, \frac{3}{5}\pi, \frac{4}{5}\pi$. **10.** $y^3-21y^2+35y-7=0$.

12. $32x^5-16x^4-32x^3+12x^2+6x-1=0$; $\frac{1}{2}$; 6.

13. $x^4-10x^2+5=0$. **14.** $\sqrt{7}$; 5.

16. $x^4-36x^3+126x^2-84x+9=0$. **21.** $\theta=\frac{\pi}{12}, \frac{5\pi}{12}$, and $\frac{9\pi}{12}$.

PAGES 55-58

1. $\theta=1°58'$. **4.** $\theta=1°58'$. **8.** $-3'$ approx.

10. (i) $-\frac{1}{6}$. (ii) 24. (iii) $\frac{1}{3}n$. (iv) $-\frac{1}{2}$.
(v) $\frac{1}{2}(\pi/180)^2$. (vi) 3. (vii) 0. (viii) 1.
(ix) e. (x) 1. (xi) 1. (xii) $-\frac{1}{84}$.
(xiii) 0. (xiv) $-\frac{1}{12}$. (xv) -3. (xvi) $-1/a$.
(xvii) 2. (xviii) 1. (xix) $exp(-\frac{1}{2}x^2)$.
(xx) 0. (xxi) 1. (xxii) $e^{-1/2}$.

11. $a=120, b=60, c=180$. **12.** $\sin x+x\cos x$.

13. $\theta^2-\frac{1}{3}\theta^4+\frac{2}{45}\theta^6-\ldots+(-1)^{n+1}.\frac{1}{2}(2\theta)^{2n}/(2n)!+\ldots$,
$1-(3/2)\theta^2+(7/8)\theta^4-\ldots+(-1)^n\{3^{2n}+3\}\theta^{2n}/4.(2n)!+\ldots$,
$\theta+(1/3)\theta^3+(2/15)\theta^5+(17/315)\theta^7+\ldots$

15. (i) $1-(1/2)\theta^2+\ldots$
$+(-1)^{n-1}\{3^{2n+1}-3\}\theta^{2n-2}/4.(2n+1)!+\ldots$
(ii) $\theta-(7/6)\theta^3+\ldots$
$+(-1)^n\{3^{2n+1}+1\}\theta^{2n+1}/4.(2n+1)!+\ldots$

PAGES 67-68

3. (i) $e^{\cos\theta}[\cos(\sin\theta)+i\sin(\sin\theta)]$.
(ii) $e^{x\alpha-y\beta}[\cos(x\beta+y\alpha)+i\sin(x\beta+y\alpha)]$.
(iii) $i\dfrac{e^\theta-e^{-\theta}}{2}$.
(iv) $\frac{1}{2}[(e^\phi+e^{-\phi})\cos\theta+i(e^\phi-e^{-\phi})\sin\theta]$.
(v) $\frac{1}{2}(y+ix)(e^\theta-e^{-\theta})/(x^2+y^2)$.
(vi) $\cos\lambda+i\sin\lambda$, where $\lambda=\frac{1}{2}(e^\theta-e^{-\theta})$.

(vii) $-\dfrac{e^{\theta}-e^{-\theta}}{e^{\theta}+e^{-\theta}}(\sin\phi-i\cos\phi)$.

(viii) $(a+ib)/(a^2+b^2)$, where $a=\frac{1}{2}(e^y+e^{-y})\cos x$ and $b=\frac{1}{2}(e^y-e^{-y})\sin x$.

6. $X=e^p\cos q$, $Y=e^p\sin q$, where

$$p=(x^2+y^2-a^2)/\{(x+a)^2+y^2\}$$

and $$q=2ay/\{(x+a)^2+y^2\}.$$

Pages 71-72.

2. $(2n+1)\pi i+\log 3$.

9. $\frac{1}{2}(4n-1)\pi i$, $-\frac{1}{2}\pi i$.

Pages 75-76

2. (i) $e^x(\cos y+i\sin y)$, where

$$x=-\tfrac{1}{2}(4n+1)\pi\beta \quad\text{and}\quad y=\tfrac{1}{2}(4n+1)\pi\alpha.$$

(ii) $\cos\{\frac{1}{2}(4n+1)\pi\}-i\sin\{\frac{1}{2}(4n+1)\pi\}$.

(iii) $\cos\theta+i\sin\theta$, where $\theta=\frac{1}{2}(4m+1)\pi\,.\exp\{-\frac{1}{2}(4n+1)\pi\}$.

3. $e^{-\tan^{-1}(4/3)}[\cos(\log 5)+i\sin(\log 5)]$.

Pages 78-80

3. $\exp[\{\log(a\sec\phi)\}^2+\phi^2]$.

12. $(A+iB)/C$, where $A=\frac{1}{4}\log(x^2+y^2)\log(\alpha^2+\beta^2)$
$+[2m\pi+\tan^{-1}(y/x)][2n\pi+\tan^{-1}(\beta/\alpha)]$,
$B=\frac{1}{2}[2m\pi+\tan^{-1}(y/x)]\log(\alpha^2+\beta^2)$
$-\frac{1}{2}[2n\pi+\tan^{-1}(\beta/\alpha)]\log(x^2+y^2)$,
$C=[\frac{1}{2}\log(\alpha^2+\beta^2)]^2+(2n\pi+\tan^{-1}(\beta/\alpha))^2$.

Pages 93-97

2. $\frac{1}{2}\log 7$. **3.** $\frac{1}{3}$.

5. (i) $\cosh x\cos y+i\sinh x\sin y$.

(ii) $[\sin(2x)-i\sinh(2y)]/[\cosh(2y)-\cos(2x)]$.

(iii) $[\sinh(2x)+i\sin(2y)]/[\cosh(2x)+\cos(2y)]$.

(iv) $2(\cos x\cosh y+i\sin x\sinh y)/(\cos 2x+\cosh 2y)$.

(v) $\frac{1}{2}[(1-\cos 2x\cosh 2y)+i\sin 2x\sinh 2y]$.

(vi) $e^{\sin\alpha\cosh\beta}.[\cos(\cos\alpha\sinh\beta)+i\sin(\cos\alpha\sinh\beta)]$.

6. $\log(\pi/2)+(\pi/2)i$, $\frac{1}{2}\log\{\frac{1}{2}(\cosh 2y+\cos 2x)\}$
$-i\tan^{-1}(\tan x\tanh y)$.

7. (i) $i\tanh^{-1}\{2y/(1+x^2+y^2)\}$; (ii) $x(1+x^2+y^2)/(x^2+y^2)$.

11. $u=\sin 2x/(\cos 2x+\cosh 2y)$,
$v=\sinh 2y/(\cos 2x+\cosh 2y)$.

29. $\sin^{-1}\sqrt{(\sin\theta)}+i\log\{\sqrt{(1+\sin\theta)}-\sqrt{(\sin\theta)}\}$.

PAGES 104-105

7. $-(\frac{1}{2})^6[\sin 7\theta+\sin 5\theta-3\sin 3\theta-3\sin\theta]$.

8. $(\frac{1}{2})^6[\sin 7\theta-3\sin 5\theta+\sin 3\theta+5\sin\theta]$.

9. $-(\frac{1}{2})^7[\cos 8\theta-4\cos 6\theta+4\cos 4\theta+4\cos 2\theta-5]$.

10. $-(\frac{1}{2})^7[\sin 8\theta+2\sin 6\theta-2\sin 4\theta-6\sin 2\theta]$.

11. $-(\frac{1}{2})^{11}[\sin 12\theta-2\sin 10\theta-4\sin 8\theta+10\sin 6\theta$
$+5\sin 4\theta-20\sin 2\theta]$.

PAGES 117-118

8. Other roots are $\cos(3\pi/7)$ and $\cos(5\pi/7)$.

11. $\{(-1)^{(n-1)/2}.\sin n\theta\}/2^{n-1}$, when n is odd; and
$\{(-1)^{n/2}(1-\cos n\theta)\}/2^{n-1}$, when n is even.

12. $(-1)^{(n-1)/2}.n\sec n\theta$, when n is odd; and 0, when n is even.

13. $n^2\operatorname{cosec}^2(n\theta)$, when n is odd; and $n^2/(1-\cos n\theta)$, when n is even.

14. $n^2/\{1-(-1)^{n/2}\cos n\theta\}$, when n is even; and $n^2\sec^2 n\theta$, when n is odd.

17. $\sqrt{2}\cos(n\pi/4)$.

19. $\frac{1}{2}+\frac{1}{2}(-1)^n\{1-2n^2\cos^2\theta+(2/3)n^2(n^2-1)\cos^4\theta+\ldots$
$+(-1)^n2^{2n-1}\cos^{2n}\theta\}$.

PAGES 118-120

1. $A=1, B=-1, C=-3, D=3$.

3. $\cos 8\theta=1-32\cos^2\theta+160\cos^4\theta-256\cos^6\theta+128\cos^8\theta$.
$\cos^8\theta=(\frac{1}{2})^7[\cos 8\theta+8\cos 6\theta+28\cos 4\theta+56\cos 2\theta+35]$.

10. $n^2\sin^2\theta-(1/3)n^2(n^2-1)\sin^4\theta+\ldots+(-1)^{n-1}2^{2n-2}\sin^{2n}\theta$.

11. $n^2\sec^2 n\theta-n$, when n is odd; and $n^2/\{(1-(-1)^{n/2}\cos n\theta\}$
$-n$, when n is even.

12. 0, when n is odd; and $n^2/2\{(-1)^{n/2}\cos n\theta - 1)\}$, when n is even.

PAGES 125-127

1. $r\sin\theta \,.\, x + \dfrac{r^2\sin 2\theta}{2!}x^2 + \ldots + \dfrac{r^n\sin n\theta}{n!}x^n + \ldots,$

where $r^2 = a^2 + b^2$, and $\tan\theta = b/a$.

2. $\theta = x\cos\alpha - \frac{1}{2}x^2\sin 2\alpha - \frac{1}{3}x^3\cos 3\alpha + \frac{1}{4}x^4\sin 4\alpha + \frac{1}{5}x^5\cos 5\alpha - \ldots$ ad inf.

8. $-2(x\cos\theta + \frac{1}{2}x^2\cos 2\theta + \frac{1}{3}x^3\cos 3\theta + \ldots)$.

9. $\cos\theta + a\cos(\theta+\phi) + \dfrac{a^2}{2!}\cos(\theta+2\phi) + \dfrac{a^2}{3!}\cos(\theta+3\phi) + \ldots$ ad inf.

11. $2(x\cos\theta - \frac{1}{3}x^3\cos 3\theta + \frac{1}{5}x^5\cos 5\theta - \ldots)$.

PAGE 131

2. $2; -1; 5.$

6. $(n+\frac{1}{4})\pi + \tan\theta - \frac{1}{3}\tan^3\theta + \frac{1}{5}\tan^5\theta - \ldots$.

PAGES 131-133

2. $\cos\theta + a\cos(\theta+\phi) + a^2\cos(\theta+2\phi) + \ldots$

3. $2(\sin\theta - \frac{1}{3}\sin 3\theta + \frac{1}{5}\sin 5\theta - \ldots)$.

PAGES 138-139

1. (i) $\sin\{\alpha + \frac{1}{2}(n-1)(\pi+\beta)\} \,.\, \sin\{\frac{1}{2}n(\pi+\beta)\} \,.\, \mathrm{cosec}\,\{\frac{1}{2}(\pi+\beta)\}$

(ii) $\frac{1}{2}\sin(2n\theta) \,.\, \mathrm{cosec}\,\theta$.

(iii) $\frac{1}{2}[n - \cos\{2\alpha + (n-1)\beta\} \,.\, \sin n\beta \,.\, \mathrm{cosec}\,\beta]$.

(iv) $\frac{1}{2}\{n + \cos(n+1)\theta \,.\, \sin n\theta \,.\, \mathrm{cosec}\,\theta\}$.

(v) $\frac{1}{4}[3\sin\{\frac{1}{2}(n+1)\alpha\} \,.\, \sin(n\alpha/2) \,.\, \mathrm{cosec}\,(\alpha/2) - \sin\{3(n+1)\alpha/2\} \,.\, \sin(3n\alpha/2) \,.\, \mathrm{cosec}\,(3\alpha/2)]$.

(vi) $\frac{1}{8}[3n - 4\cos(n+1)\alpha \,.\, \sin n\alpha \,.\, \mathrm{cosec}\,\alpha + \cos 2(n+1)\alpha \,.\, \sin 2n\alpha \,.\, \mathrm{cosec}\,2\alpha]$.

(vii) $\frac{1}{2}[\cos 2(n+1)\theta \,.\, \sin 2n\theta \,.\, \mathrm{cosec}\,2\theta + n\cos 2\theta]$.

3. $\sin\frac{1}{2}(n+1)\alpha \,.\, \sin\frac{1}{2}n\alpha \,.\, \mathrm{cosec}\,\frac{1}{2}\alpha\,;\ \frac{1}{2}(n+1)$.

6. $\frac{1}{2}[n + \sin(n\pi/2)\cos\{2\alpha + \frac{1}{2}(n-1)\pi\}]$.

7. 0.

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1. $\log\{2\cos(\theta/2)\}$.
2. $e^{\cos\alpha} . \sin(\sin\alpha)$.
3. $4\sin\alpha/(5-4\cos\alpha)$.
4. $\tan^{-1}\{x\sin\theta/(1+x\cos\theta)\}$, except when $x=1$, and $\theta=(2n+1)\pi$.
5. $\frac{1}{4}\log\{(1+2x\cos\alpha+x^2)/(1-2x\cos\alpha+x^2)\}$.
6. $(1-x\cos\theta)/(1-2x\cos\theta+x^2)$.
7. $\cos(\theta/4)/\sqrt{\{2\cos(\theta/2)\}}$.
8. $\pi/4, -\pi/4$, or 0, according as $\cos\theta$ is positive, negative, or zero.
9. $\cos(\cos\theta)\sinh(\sin\theta)$.
10. $\sin(\pi/4+\alpha/2)/\sqrt{(2\sin\alpha)}$, except when $\alpha=n\pi$.
11. $\sin\{(\pi-\alpha)/4\}/\sqrt{(2\sin\frac{1}{2}\alpha)}$.
12. $\cos(x\sin\theta) . \cosh(x\cos\theta)$.
13. $\tan^{-1}\{x\sin\alpha/(1-x\cos\alpha)\}$.
14. $e^{-\cos\alpha\cos\beta} . \cos(\cos\alpha\sin\beta)$.
15. 1.
16. $\sin(\sin\theta\cos\theta) . \exp(\cos^2\theta)$.
17. $e^{\sin\theta\cos\theta} . \cos(\theta+\sin^2\theta)$.
18. $x\cos\theta/(1-x\cos\theta)-\log(1-x\cos\theta)$.
19. $\frac{1}{8}\{2\sqrt{3}\log(2+\sqrt{3})-\pi\}$.

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1. $\sinh\frac{1}{2}(n+1)\theta . \sinh\frac{1}{2}n\theta . \operatorname{cosech}\frac{1}{2}\theta$.
2. $\cosh\{x+\frac{1}{2}(n-1)y\} . \sinh\frac{1}{2}ny . \operatorname{cosech}\frac{1}{2}y$.
3. $\frac{1}{2}n+\frac{1}{2}\cosh\{2\alpha+(n+1)\beta\}\sinh n\beta \operatorname{cosech}\beta$.
4. $\frac{1}{2}\{\exp(e^x)+\exp(e^{-x})\}$.
5. $\frac{1}{2}\{\exp(xe^\theta)+\exp(xe^{-\theta})\}-1$.

PAGES 150-152

1. $\cot(\frac{1}{2}x)-\cot(2^{n-1}x)$.
2. $\cot\theta-2^n\cot(2^n\theta)$.
3. $\operatorname{cosec}\alpha\{\tan(n+1)\alpha-\tan\alpha\}$.
4. $\tan^{-1}(n+1)x-\tan^{-1}x$.
5. $\frac{1}{4}\pi$.
6. $\tan^{-1}\{n/(n+2)\}$.
7. $\frac{1}{4}\{3^n\sin(\theta/3^n)-\sin\theta\}$.

8. $\frac{1}{4}\operatorname{cosec}\frac{1}{2}\theta\,.\,\{\sec\frac{1}{2}(2n+1)\theta-\sec\frac{1}{2}\theta\}$.

9. $\frac{1}{8}[3^n\tan(3^n\theta)-\tan\theta]$.

10. $\tan x$. **11.** $\cot^{-1}(\frac{1}{2}a)-\cot^{-1}\{\frac{1}{2}(n+1)a\}$.

12. $\frac{1}{4}\pi$. **13.** $\cot^{-1}(1/2)$.

14. $(1/2^{n-1})\tan(2^n\alpha)-2\tan\alpha$. **15.** $\operatorname{cosec}^2\theta-2^n\operatorname{cosec}^2(2^n\theta)$.

16. $\frac{1}{2}\sin\theta\{\cot(\theta/2^{n-1})-\cot(\theta/2)\}$.

Pages 154-155

2. $\sin^{-1}\{\sqrt{2}\sin(\theta/2)\}$; $\sin^{-1}\{\sqrt{2}\cos(\theta/2)\}$.

Pages 155-158

1. $\frac{1}{2}\sin\frac{1}{2}\{2n\alpha+(n+1)\beta\}\,.\,\sin\frac{1}{2}n(2\alpha+\beta)\,.\,\operatorname{cosec}\frac{1}{2}(2\alpha+\beta)$
$-\frac{1}{2}\sin\frac{1}{2}\{2n\alpha-(n+1)\beta\}\,.\,\sin\frac{1}{2}n(2\alpha-\beta)\,.\,\operatorname{cosec}\frac{1}{2}(2\alpha-\beta)$.

2. $1-(1-\cos\theta)\log\{2\sin(\frac{1}{2}\theta)\}-\frac{1}{2}(\pi-\theta)\sin\theta$, unless θ is a multiple of 2π, when the sum is 1.

3. $\frac{1}{4}[\sin\frac{1}{2}n\theta\,.\,\{2\sin\frac{1}{2}(n+3)\theta+\sin\frac{1}{4}(n-1)\theta\}\operatorname{cosec}\frac{1}{2}\theta$
$-\sin\frac{1}{2}(3n+5)\,\theta\,.\,\sin\frac{3}{2}n\theta\,.\,\operatorname{cosec}\frac{3}{2}\theta]$.

5. $(1/m)\,\{2\sin(\frac{1}{2}\theta)\}^m\sin\{\frac{1}{2}m(\pi-\theta)\}$.

6. (i) $\frac{1}{4}\log\{(1+2c\sin\alpha+c^2)/(1-2c\sin\alpha+c^2)\}$.
(ii) $\frac{1}{2}[(\cosh(\cos\alpha)\sin(\sin\alpha)+\cos(\cos\alpha)\sinh(\sin\alpha)]$.

7. $\cot\beta\{\tan(\alpha+n\beta)-\tan\alpha\}-n$.

8. $\log\dfrac{\sin 2\theta}{2^n\sin(\theta/2^{n-1})}$, $\log\dfrac{\sin 2\theta}{2\theta}$.

9. 0. **10.** $\frac{1}{2}(\cot\theta-3^n\cot 3^n\theta)$. **11.** $\frac{3}{8}n$.

12. $\operatorname{cosec}(\theta+\frac{1}{2}\pi)\,.\,\{\tan(n+1)(\theta+\frac{1}{2}\pi)-\tan(\theta+\frac{1}{2}\pi)\}$.

13. $\exp\{e^{\cos\theta}\cos(\sin\theta)\}\,.\,\cos\{e^{\cos\theta}\sin(\sin\theta)\}$.

14. $\sqrt{2}\,.\,\sin\frac{1}{4}\{\pi+(n+1)x\}\,.\,\sin\frac{1}{4}nx\,.\,\operatorname{cosec}\frac{1}{4}x$.

15. $\tan(2^n\theta)-\tan\theta$.

16. (i) $\cos\{\alpha+\frac{1}{2}(n-1)(\beta+\pi)\}\sin\frac{1}{2}n(\beta+\pi)\,.\,\operatorname{cosec}\frac{1}{2}(\beta+\pi)$.
(ii) $\cosh\{\alpha+\frac{1}{2}(n-1)\beta\}\,.\,\sinh\frac{1}{2}n\beta\,.\,\operatorname{cosech}\frac{1}{2}\beta$.

17. $(1/32)\,[2n+\cos(n+1)x\,.\,\sin nx\,.\,\operatorname{cosec}x$
$-2\cos 2(n+1)x\,.\,\sin 2nx\,.\,\operatorname{cosec}2x$
$-\cos 3(n+1)x\,.\,\sin 3nx\,.\,\operatorname{cosec}3x]$.

18. $\cot^{-1}(2^{n+1}+2^{-n})$; $\cot^{-1}(2)$.

19. $\frac{1}{2}\log\{\sin\frac{1}{2}(\alpha+\beta)\,.\,\operatorname{cosec}\frac{1}{2}(\alpha-\beta)\}$, except when $\alpha\pm\beta$ is a multiple of 2π.

20. $\frac{1}{4}\pi$ or 0 according as $\theta >$ or $< \phi$.

21. $\theta + \sin\theta\cos\theta$.

22. $\frac{1}{2}\sin 2\theta + (-\frac{1}{2})^{n+1}\sin 2^{n+1}\theta$.

23. $\cos(\cos\theta) \,.\, \cosh(\sin\theta)$. **24.** $\frac{1}{2}\log(\text{cosec}\, x)$.

25. $\frac{1}{8}\{(\frac{1}{3})^{n-1}\tan 3^n\theta\tan\theta\}$. **26.** $1/\pi$.

30. $20A_1 \,.\, \cos(\frac{1}{2}\theta - \pi/2n) \,.\, \text{cosec}\,(\pi/2n)$.

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2. $(x-1)(x+1)(x^2+1)(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)$.

3. $(x+1)\prod_{r=0}^{2}[x^2-2x\cos\{(2r+1)\pi/7\}+1]$.

4. $\prod_{r=0}^{r=5}[x^2-2x\cos\{(2r+1)\pi/12\}+1]$.

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21. $(1/\pi)\sinh\pi$.

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10. $6554 + i\,4449$; or $r^7(\cos 7\theta + i\sin 7\theta)$, where $r = \sqrt{(13)}$, $\cos\theta = 2/\sqrt{(13)}$, and $\sin\theta = 3/\sqrt{(13)}$.

12. $\frac{1}{2}(1-i)\{\cot(n\pi/5)-1\}$. **13.** $\theta = (2n\pi+\pi)/10$; $\cot(\pi/10) = \sqrt{\{5+2\sqrt{5}\}}$.

22. $\alpha = (\sqrt{13}-1)/4$; $\beta = -(\sqrt{13}+1)/4$.

23. $a=5$, $b=-4$, $V=1$.

30. $\sinh 12x - \sinh 8x - \sinh 4x$. **41.** $2^{n-1}(n!)$.

43. $\cos\theta + x\cos 2\theta + x^2\cos 3\theta + \ldots$.

51. $4\sinh(\cos\theta)\cos(\sin\theta) + \cos\theta\cosh(\cos\theta)\cos(\sin\theta) - \sin\theta\sinh(\cos\theta)\sin(\sin\theta)$.

53. $\frac{1}{2}n(\cos\alpha \mp i\sin\alpha)$ for the two cases, viz. $x = \cos(2\pi/n) \pm \sin(2\pi/n)$.

54. $r^n\cos(\theta+n\lambda)$, where $r^2 = 1+2x\cos\phi+x^2$, and $\tan\lambda = x\sin\phi/(1+x\cos\phi)$.

60. $\pi^3\sqrt{3}/54$.